Monotone Convergence Theorem — completeness at work

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Practice problems

1. Show that the sequence $a_n = \sum_{k=1}^{n} \frac{1}{k^2}$ is convergent using the Monotone Convergence Theorem.
2. Let $a_1 = \sqrt{2}$ and $a_{n+1} = \sqrt{2 + a_n}$. Prove the sequence converges and find its limit.
3. Does the sequence $a_n = \frac{n}{n+1}$ converge? Justify using the Monotone Convergence Theorem.

Solutions

1. Show that the sequence $a_n = \sum_{k=1}^{n} \frac{1}{k^2}$ is convergent using the Monotone Convergence Theorem.

   - Step 1: Check monotonicity. $a_{n+1} - a_n = \frac{1}{(n+1)^2} > 0$, so the sequence is strictly increasing.
   - Step 2: Check boundedness. We know $\frac{1}{k^2} \leq \frac{1}{k(k-1)}$ for $k \geq 2$.
   - Step 3: The sum $\sum_{k=2}^{n} \frac{1}{k(k-1)}$ is a telescoping sum that equals $1 - \frac{1}{n}$.
   - Step 4: Thus $a_n \leq 1 + (1 - \frac{1}{n}) < 2$. The sequence is bounded above by 2.
   - Step 5: Since it is increasing and bounded above, it converges by the MCT.

   Answer: The sequence converges because it is monotonically increasing and bounded above by 2.

2. Let $a_1 = \sqrt{2}$ and $a_{n+1} = \sqrt{2 + a_n}$. Prove the sequence converges and find its limit.

   - Step 1: Prove $a_n < 2$ by induction. $a_1 = \sqrt{2} < 2$. If $a_n < 2$, then $a_{n+1} = \sqrt{2 + a_n} < \sqrt{4} = 2$.
   - Step 2: Prove $a_{n+1} > a_n$. $a_{n+1}^2 - a_n^2 = 2 + a_n - a_n^2 = (2 - a_n)(1 + a_n)$. Since $a_n < 2$ and $a_n > 0$, this is positive.
   - Step 3: By MCT, the limit L exists. Solve $L = \sqrt{2 + L}$.
   - Step 4: $L^2 - L - 2 = 0 \implies (L - 2)(L + 1) = 0$. Since $a_n > 0$, $L = 2$.

   Answer: L = 2

3. Does the sequence $a_n = \frac{n}{n+1}$ converge? Justify using the Monotone Convergence Theorem.

   - Step 1: Monotonicity. $a_n = 1 - \frac{1}{n+1}$. As n increases, $\frac{1}{n+1}$ decreases, so $1 - \frac{1}{n+1}$ increases. It is monotonically increasing.
   - Step 2: Boundedness. Since $n < n + 1$, $a_n < 1$ for all n. It is bounded above by 1.
   - Step 3: By MCT, the sequence converges to its supremum, which is 1.

   Answer: Yes, it converges to 1.
