The squeeze theorem — proved

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Practice problems

1. Use the Squeeze Theorem to find the limit of the sequence $b_n = \frac{\cos(n)}{n^2}$ as $n \to \infty$.
2. Let $a_n = \frac{1}{n+1}$ and $c_n = \frac{1}{n-1}$ for $n \geq 2$. If $a_n \leq b_n \leq c_n$, what is the limit of b_n?
3. Prove that the sequence $b_n = \frac{(-1)^n}{n}$ converges to 0 using the Squeeze Theorem.

Solutions

1. Use the Squeeze Theorem to find the limit of the sequence $b_n = \frac{\cos(n)}{n^2}$ as $n \to \infty$.

   - Step 1: Identify the bounds of the cosine function: $-1 \leq \cos(n) \leq 1$.
   - Step 2: Divide the entire inequality by n^2, which is always positive: $-\frac{1}{n^2} \leq \frac{\cos(n)}{n^2} \leq \frac{1}{n^2}$.
   - Step 3: Find the limits of the guards: $\lim_{n\to\infty} -\frac{1}{n^2} = 0$ and $\lim_{n\to\infty} \frac{1}{n^2} = 0$.
   - Step 4: Since both guards converge to 0, by the Squeeze Theorem, the middle sequence must also converge to 0.

   Answer: 0

2. Let $a_n = \frac{1}{n+1}$ and $c_n = \frac{1}{n-1}$ for $n \geq 2$. If $a_n \leq b_n \leq c_n$, what is the limit of b_n?

   - Step 1: Calculate the limit of the lower guard: $\lim_{n\to\infty} \frac{1}{n+1} = 0$.
   - Step 2: Calculate the limit of the upper guard: $\lim_{n\to\infty} \frac{1}{n-1} = 0$.
   - Step 3: Since $a_n \to 0$ and $c_n \to 0$, and b_n is squeezed between them, b_n must also converge to 0.

   Answer: 0

3. Prove that the sequence $b_n = \frac{(-1)^n}{n}$ converges to 0 using the Squeeze Theorem.

   - Step 1: Note that $(-1)^n$ is always either -1 or 1. Thus, $-1 \leq (-1)^n \leq 1$.
   - Step 2: Divide by n: $-\frac{1}{n} \leq \frac{(-1)^n}{n} \leq \frac{1}{n}$.
   - Step 3: The lower guard $a_n = -\frac{1}{n}$ converges to 0.
   - Step 4: The upper guard $c_n = \frac{1}{n}$ converges to 0.
   - Step 5: By the Squeeze Theorem, $b_n \to 0$.

   Answer: 0
