Algebra of limits — proved

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Practice problems

1. Given that $a_n \to L$ and $b_n \to M$, prove using the ϵ-N definition that $a_n + b_n \to L + M$.
2. Use the product rule for limits to find the limit of the sequence $s_n = (1 + 1/n)(2 + 1/n^2)$.
3. Suppose $a_n \to L$. Use the product rule to prove that $a_n^2 \to L^2$.

Solutions

1. Given that $a_n \to L$ and $b_n \to M$, prove using the ϵ-N definition that $a_n + b_n \to L + M$.

   - Let $\epsilon > 0$.
   - Since $a_n \to L$, there exists N_1 such that $|a_n - L| < \epsilon/2$ for $n \geq N_1$.
   - Since $b_n \to M$, there exists N_2 such that $|b_n - M| < \epsilon/2$ for $n \geq N_2$.
   - Let $N = \max(N_1, N_2)$. For $n \geq N$, $|(a_n + b_n) - (L + M)| = |(a_n - L) + (b_n - M)| \leq |a_n - L| + |b_n - M| < \epsilon/2 + \epsilon/2 = \epsilon$.

   Answer: The sum converges to $L + M$.

2. Use the product rule for limits to find the limit of the sequence $s_n = (1 + 1/n)(2 + 1/n^2)$.

   - Identify $a_n = 1 + 1/n$ and $b_n = 2 + 1/n^2$.
   - Note that $a_n \to 1$ and $b_n \to 2$.
   - By the product rule, $\lim s_n = (\lim a_n)(\lim b_n) = 1 \times 2 = 2$.

   Answer: 2

3. Suppose $a_n \to L$. Use the product rule to prove that $a_n^2 \to L^2$.

   - Let $b_n = a_n$. We are given $a_n \to L$ and $b_n \to L$.
   - The sequence a_n^2 is the product $a_n b_n$.
   - By the product rule, $\lim(a_n b_n) = (\lim a_n)(\lim b_n) = L \times L = L^2$.

   Answer: The sequence converges to L^2.
