Uniqueness of limits; bounded sequences

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Practice problems

1. Suppose $a_n \to 0$. Prove that $|a_n| \to 0$.
2. Give an example of a sequence that is bounded but does not converge, and explain why it fails to converge.
3. Prove that if $a_n \to L$ and $b_n \to M$, then the sequence $c_n = a_n - b_n$ converges to $L - M$.

Solutions

1. Suppose $a_n \to 0$. Prove that $|a_n| \to 0$.

   - We want to show that for any $\epsilon > 0$, there exists N such that $||a_n| - 0| < \epsilon$ for all $n \geq N$.
   - Note that $||a_n| - 0| = ||a_n|| = |a_n|$.
   - Since $a_n \to 0$, for any $\epsilon > 0$, there exists N such that $|a_n - 0| < \epsilon$ for all $n \geq N$.
   - This simplifies to $|a_n| < \epsilon$ for all $n \geq N$.
   - Thus, $||a_n| - 0| < \epsilon$ for all $n \geq N$, so $|a_n| \to 0$.

   Answer: The sequence $|a_n|$ converges to 0.

2. Give an example of a sequence that is bounded but does not converge, and explain why it fails to converge.

   - Consider $a_n = (-1)^n$.
   - The sequence is bounded because $|a_n| = 1$ for all n, so we can pick $M = 1$.
   - To show it diverges, assume it converges to L. Then for $\epsilon = 0.5$, there must be an N such that $|a_n - L| < 0.5$ for all $n \geq N$.
   - For $n = N$ and $n = N + 1$, we have $|a_N - L| < 0.5$ and $|a_{N+1} - L| < 0.5$.
   - But $|a_N - a_{N+1}| = |1 - (-1)| = 2$.
   - By triangle inequality, $2 = |a_N - a_{N+1}| \leq |a_N - L| + |L - a_{N+1}| < 0.5 + 0.5 = 1$.
   - Since $2 < 1$ is a contradiction, the sequence diverges.

   Answer: $a_n = (-1)^n$ is bounded but diverges due to oscillation.

3. Prove that if $a_n \to L$ and $b_n \to M$, then the sequence $c_n = a_n - b_n$ converges to $L - M$.

   - We want to show $|(a_n - b_n) - (L - M)| < \epsilon$.
   - Rewrite the expression as $|(a_n - L) - (b_n - M)|$.
   - Apply the triangle inequality: $|(a_n - L) - (b_n - M)| \leq |a_n - L| + |b_n - M|$.
   - Since $a_n \to L$, choose N_1 such that $|a_n - L| < \epsilon/2$ for $n \geq N_1$.
   - Since $b_n \to M$, choose N_2 such that $|b_n - M| < \epsilon/2$ for $n \geq N_2$.
   - For $n \geq \max(N_1, N_2)$, the sum is less than $\epsilon/2 + \epsilon/2 = \epsilon$.

   Answer: $c_n \to L - M$.
