

First convergence proofs — the template, slowly

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Practice problems

1. Prove that the sequence $a_n = \frac{3}{n+1}$ converges to 0.
2. Prove that the sequence $a_n = \frac{n}{n+5}$ converges to 1.
3. Prove that the sequence $a_n = \frac{4n-1}{2n+3}$ converges to 2.

Solutions

1. Prove that the sequence $a_n = \frac{3}{n+1}$ converges to 0.

- Scratch work: $|\frac{3}{n+1} - 0| < \epsilon \implies \frac{3}{n+1} < \epsilon \implies n+1 > \frac{3}{\epsilon} \implies n > \frac{3}{\epsilon} - 1$.
- Let $\epsilon > 0$. Choose $N = \lceil \frac{3}{\epsilon} - 1 \rceil$ (or 1 if this is non-positive).
- Let $n \geq N$. Then $n \geq \frac{3}{\epsilon} - 1$.
- Adding 1 gives $n+1 \geq \frac{3}{\epsilon}$, so $\frac{1}{n+1} \leq \frac{\epsilon}{3}$.
- Multiplying by 3 gives $\frac{3}{n+1} \leq \epsilon$. Thus $|a_n - 0| \leq \epsilon$.

Answer: The sequence converges to 0.

2. Prove that the sequence $a_n = \frac{n}{n+5}$ converges to 1.

- Scratch work: $|\frac{n}{n+5} - 1| = |\frac{n-(n+5)}{n+5}| = |\frac{-5}{n+5}| = \frac{5}{n+5}$.
- We want $\frac{5}{n+5} < \epsilon \implies n+5 > \frac{5}{\epsilon} \implies n > \frac{5}{\epsilon} - 5$.
- Let $\epsilon > 0$. Choose $N = \max(1, \lceil \frac{5}{\epsilon} - 5 \rceil)$.
- For $n \geq N$, $n+5 \geq \frac{5}{\epsilon}$, so $\frac{5}{n+5} \leq \epsilon$.
- Therefore $|a_n - 1| \leq \epsilon$.

Answer: The sequence converges to 1.

3. Prove that the sequence $a_n = \frac{4n-1}{2n+3}$ converges to 2.

- Scratch work: $|\frac{4n-1}{2n+3} - 2| = |\frac{4n-1-2(2n+3)}{2n+3}| = |\frac{4n-1-4n-6}{2n+3}| = \frac{7}{2n+3}$.
- We want $\frac{7}{2n+3} < \epsilon \implies 2n+3 > \frac{7}{\epsilon} \implies 2n > \frac{7}{\epsilon} - 3 \implies n > \frac{7}{2\epsilon} - 1.5$.
- Let $\epsilon > 0$. Choose $N = \max(1, \lceil \frac{7}{2\epsilon} - 1.5 \rceil)$.
- For $n \geq N$, $2n+3 \geq \frac{7}{\epsilon}$, so $\frac{7}{2n+3} \leq \epsilon$.
- Therefore $|a_n - 2| \leq \epsilon$.

Answer: The sequence converges to 2.