Convergence: the eps-N definition — the adversarial game

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Practice problems

1. Use the epsilon-N definition to prove that the sequence $a_n = \frac{5}{n+2}$ converges to 0.
2. Prove that the sequence $a_n = \frac{n}{n+1}$ converges to 1 using the epsilon-N definition.
3. Show that the sequence $a_n = (-1)^n$ does not converge to 0 by using the negation of the epsilon-N definition.

Solutions

1. Use the epsilon-N definition to prove that the sequence $a_n = \frac{5}{n+2}$ converges to 0.

   - Let $\epsilon > 0$. We want $|\frac{5}{n+2} - 0| < \epsilon$.
   - This simplifies to $\frac{5}{n+2} < \epsilon$.
   - Rearranging gives $n + 2 > \frac{5}{\epsilon}$, so $n > \frac{5}{\epsilon} - 2$.
   - Choose N to be the smallest integer greater than $\frac{5}{\epsilon} - 2$.
   - Then for all $n \geq N$, we have $n > \frac{5}{\epsilon} - 2$, which implies $|a_n - 0| < \epsilon$.

   Answer: The sequence converges to 0 with $N = \lceil \frac{5}{\epsilon} - 2 \rceil$ (or any larger integer).

2. Prove that the sequence $a_n = \frac{n}{n+1}$ converges to 1 using the epsilon-N definition.

   - Let $\epsilon > 0$. We want $|\frac{n}{n+1} - 1| < \epsilon$.
   - Simplify the expression: $|\frac{n-(n+1)}{n+1}| = |\frac{-1}{n+1}| = \frac{1}{n+1}$.
   - We need $\frac{1}{n+1} < \epsilon$, which means $n + 1 > \frac{1}{\epsilon}$, or $n > \frac{1}{\epsilon} - 1$.
   - Choose $N = \lceil \frac{1}{\epsilon} \rceil$.
   - For $n \geq N$, $n + 1 > \frac{1}{\epsilon}$, so $|a_n - 1| < \epsilon$.

   Answer: The sequence converges to 1 with $N = \lceil \frac{1}{\epsilon} \rceil$.

3. Show that the sequence $a_n = (-1)^n$ does not converge to 0 by using the negation of the epsilon-N definition.

   - To prove divergence to 0, we need to find one $\epsilon > 0$ such that for every N, there is some $n \geq N$ where $|a_n - 0| \geq \epsilon$.
   - Let $\epsilon = 0.5$.
   - For any N, we can pick n to be any integer greater than or equal to N.
   - Since $|a_n| = |(-1)^n| = 1$ for all n, we have $|a_n - 0| = 1$.
   - Since $1 \geq 0.5$, the condition $|a_n - 0| \geq \epsilon$ is always satisfied for every $n \geq N$.

   Answer: The sequence does not converge to 0 because for $\epsilon = 0.5$, $|a_n - 0| \geq \epsilon$ for all n.
