# Module 1 review + self-test

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## Practice problems

1. Find the supremum and infimum of the set $S = \{\sqrt{n^2+1} - n : n \in \mathbb{N}\}$. Does the set contain its supremum or infimum?
2. Use mathematical induction to prove that for all $n \in \mathbb{N}$, $2^n > n$.
3. Prove that the set of all irrational numbers in the interval $(0, 1)$ is uncountable.

## Solutions

1. Find the supremum and infimum of the set $S = \{\sqrt{n^2+1} - n : n \in \mathbb{N}\}$. Does the set contain its supremum or infimum?

   - Rewrite the expression by multiplying by the conjugate: $\sqrt{n^2+1} - n = \frac{(\sqrt{n^2+1}-n)(\sqrt{n^2+1}+n)}{\sqrt{n^2+1}+n} = \frac{1}{\sqrt{n^2+1}+n}$.
   - For $n = 1$, the value is $1/(\sqrt{2}+1) \approx 0.414$. As $n$ increases, the denominator increases, so the values decrease.
   - The maximum value occurs at $n = 1$, so $\sup S = \sqrt{2} - 1$. Since this value is in the set, it is also the maximum.
   - As $n \to \infty$, the expression $1/(\sqrt{n^2+1}+n)$ approaches 0. Since the values are always positive, the infimum is 0.
   - Since $1/(\sqrt{n^2+1}+n)$ is never 0 for any $n \in \mathbb{N}$, the infimum is not in the set.

   **Answer:** \sup S = \sqrt2-1 \text (contained), \inf S = 0 \text (not contained)

2. Use mathematical induction to prove that for all $n \in \mathbb{N}$, $2^n > n$.

   - Base case: For $n = 1$, $2^1 = 2$ and $2 > 1$. The base case holds.
   - Inductive step: Assume $2^k > k$ for some $k \in \mathbb{N}$.
   - We want to show $2^{k+1} > k + 1$.
   - Note that $2^{k+1} = 2 \cdot 2^k$. By the inductive hypothesis, $2 \cdot 2^k > 2k$.
   - Since $k \geq 1$, we know $2k = k + k \geq k + 1$.
   - Therefore, $2^{k+1} > k + 1$, completing the induction.

   **Answer:** Proven by induction.

3. Prove that the set of all irrational numbers in the interval $(0, 1)$ is uncountable.

   - Let $I = (0, 1)$ be the set of all real numbers in the interval, and let $\mathbb{Q}_I$ be the set of all rational numbers in that interval.
   - We know from the course that $I$ is uncountable ($|I| = c$).
   - We also know that the set of all rationals $\mathbb{Q}$ is countable, so any subset $\mathbb{Q}_I$ must also be countable.
   - The set of irrationals in the interval is $S = I \setminus \mathbb{Q}_I$.
   - If $S$ were countable, then $I = S \cup \mathbb{Q}_I$ would be the union of two countable sets, which must be countable.
   - This contradicts the fact that $I$ is uncountable. Therefore, $S$ must be uncountable.

   **Answer:** The set of irrationals is uncountable.