Decimal expansions justified; 0.999... = 1 forever settled

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Practice problems

1. Use the geometric series formula to find the exact fraction represented by the repeating decimal $0.727272\ldots$.
2. Prove that $0.4999\cdots = 0.5$ by expressing the left side as a sum of a constant and a geometric series.
3. Consider the decimal $0.123123\ldots$. Find its value as a fraction and explain why the sequence of partial sums S_k must converge to this value.

Solutions

1. Use the geometric series formula to find the exact fraction represented by the repeating decimal $0.727272\ldots$.

   - Identify the repeating block as 72 with a period of 2.
   - Write the expansion as a series: $\sum_{n=1}^{\infty} \frac{72}{100^n}$.
   - Identify the first term $a = \frac{72}{100}$ and the ratio $r = \frac{1}{100}$.
   - Apply the formula $S = \frac{a}{1-r} = \frac{72/100}{1-1/100} = \frac{72/100}{99/100}$.
   - Simplify the fraction $\frac{72}{99}$ by dividing by 9.

   Answer: $\frac{8}{11}$

2. Prove that $0.4999\cdots = 0.5$ by expressing the left side as a sum of a constant and a geometric series.

   - Write $0.4999\ldots$ as $0.4 + 0.0999\ldots$.
   - Express $0.0999\ldots$ as a series: $\sum_{n=2}^{\infty} \frac{9}{10^n}$.
   - The first term is $a = \frac{9}{100}$ and the ratio is $r = \frac{1}{10}$.
   - Calculate the sum: $S = \frac{9/100}{1-1/10} = \frac{9/100}{9/10} = \frac{1}{10} = 0.1$.
   - Add the constant: $0.4 + 0.1 = 0.5$.

   Answer: 0.5

3. Consider the decimal $0.123123\ldots$. Find its value as a fraction and explain why the sequence of partial sums S_k must converge to this value.

   - The repeating block is 123 with period 3.
   - The series is $\sum_{n=1}^{\infty} \frac{123}{1000^n}$.
   - Using the formula $S = \frac{123/1000}{1-1/1000} = \frac{123}{999}$.
   - Simplify by dividing by 3: $\frac{41}{333}$.
   - The sequence S_k converges because it is monotonic increasing and bounded above by 1, and the limit of the geometric series is exactly $\frac{41}{333}$.

   Answer: $\frac{41}{333}$
