The Cantor set — infinitely much of nothing

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Practice problems

1. Determine if the number $\frac{7}{27}$ is an element of the Cantor set C.
2. Prove that the endpoints of the removed intervals, such as $\frac{1}{3}$ and $\frac{2}{3}$, are always in the Cantor set.
3. If we removed the middle fifth instead of the middle third at each step, would the resulting set still be uncountable?

Solutions

1. Determine if the number $\frac{7}{27}$ is an element of the Cantor set C.

   - Convert $\frac{7}{27}$ to base three.
   - $\frac{7}{27} = \frac{6}{27} + \frac{1}{27} = \frac{2}{9} + \frac{1}{27}$.
   - In base three, this is $0 \cdot 3^{-1} + 2 \cdot 3^{-2} + 1 \cdot 3^{-3}$, which is $(0.021)_3$.
   - Since the expansion contains the digit 1, the number is removed during the third step of the construction.

   Answer: No, $\frac{7}{27} \notin C$.

2. Prove that the endpoints of the removed intervals, such as $\frac{1}{3}$ and $\frac{2}{3}$, are always in the Cantor set.

   - An endpoint is created at some step n and is never part of an open middle third removed in any step $k > n$.
   - For example, $\frac{1}{3}$ is the left endpoint of the first removed interval $(\frac{1}{3}, \frac{2}{3})$.
   - Since we only remove open intervals, the endpoints themselves are never removed.
   - Therefore, all endpoints remain in the intersection of all C_n.

   Answer: Endpoints are never removed because only open intervals are deleted.

3. If we removed the middle fifth instead of the middle third at each step, would the resulting set still be uncountable?

   - Removing the middle fifth leaves two intervals of length $\frac{2}{5}$ each.
   - The number of intervals at step n would be 2^n.
   - The points remaining would be those that can be represented in base five using only the digits $0, 1, 3, 4$ (avoiding the middle digit 2).
   - This still creates a mapping to the set of all infinite sequences of a finite alphabet with at least two symbols.
   - By Cantor's diagonal argument, this set is uncountable.

   Answer: Yes, it would still be uncountable.
