Cardinality II — Cantor's diagonal

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Practice problems

1. Prove that the set S of all infinite sequences of digits from the set $\{0, 1, 2\}$ is uncountable.
2. If a set A is uncountable and A is a subset of B, what can we conclude about the cardinality of B?
3. Prove that the set of irrational numbers in the interval $(0, 1)$ is uncountable.

Solutions

1. Prove that the set S of all infinite sequences of digits from the set $\{0, 1, 2\}$ is uncountable.

   - Assume S is countable and list the sequences as $s_1, s_2, s_3, \ldots$.
   - Construct a new sequence $x = (x_1, x_2, x_3, \ldots)$ where x_n is chosen to be different from the n-th digit of s_n.
   - For example, if the n-th digit of s_n is 0, let $x_n = 1$; otherwise, let $x_n = 0$.
   - Since x differs from every s_n at the n-th position, x is not in the list.
   - This contradicts the assumption that the list was exhaustive, so S is uncountable.

   Answer: The set S is uncountable by Cantor's diagonal argument.

2. If a set A is uncountable and A is a subset of B, what can we conclude about the cardinality of B?

   - Since $A \subseteq B$, the cardinality of B must be at least as large as the cardinality of A.
   - We are given that A is uncountable, meaning $|A| > \aleph_0$.
   - Therefore, $|B| \geq |A| > \aleph_0$.

   Answer: B must also be uncountable.

3. Prove that the set of irrational numbers in the interval $(0, 1)$ is uncountable.

   - Let $I = (0, 1)$ be the set of all real numbers in the interval, and Q_I be the set of rational numbers in that interval.
   - We know from today's lesson that $|I| = c$ (uncountable) and from yesterday that $|Q_I| = \aleph_0$ (countable).
   - The set of irrationals Irr_I is the complement $I \setminus Q_I$.
   - If Irr_I were countable, then $I = Irr_I \cup Q_I$ would be the union of two countable sets, which is countable.
   - But I is uncountable, so Irr_I must be uncountable.

   Answer: The set of irrationals is uncountable.
