# Cardinality I — countable infinity; listing the rationals

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## Practice problems

1. Prove that the set of odd natural numbers $O = \{1, 3, 5, \dots\}$ is countable by defining a bijection $f : \mathbb{N} \to O$.
2. Is the set of all finite strings of letters from the English alphabet countable? Explain your reasoning.
3. Prove that the union of two countable sets $A$ and $B$ is also countable.

## Solutions

1. Prove that the set of odd natural numbers $O = \{1, 3, 5, \dots\}$ is countable by defining a bijection $f : \mathbb{N} \to O$.

   - We need a function that maps 1 to 1, 2 to 3, 3 to 5, and so on.
   - The formula for the $n$-th odd number is $2n - 1$.
   - Check for injection: if $2n - 1 = 2m - 1$, then $2n = 2m$, so $n = m$.
   - Check for surjection: for any odd number $k$, $n = (k + 1)/2$ is a natural number such that $f(n) = k$.

   **Answer:** f(n) = 2n - 1

2. Is the set of all finite strings of letters from the English alphabet countable? Explain your reasoning.

   - We can list all strings of length 1 alphabetically.
   - Then list all strings of length 2 alphabetically.
   - Then length 3, and so on.
   - Since there are only finitely many strings of any fixed length $L$, we will eventually reach any given string in a finite number of steps.

   **Answer:** Yes, it is countable.

3. Prove that the union of two countable sets $A$ and $B$ is also countable.

   - Let $A = \{a_1, a_2, \dots\}$ and $B = \{b_1, b_2, \dots\}$ be the listings for $A$ and $B$.
   - We can create a new list by interleaving them: $a_1, b_1, a_2, b_2, a_3, b_3, \dots$.
   - If an element appears in both sets, we simply skip it the second time it appears to maintain a bijection.
   - This process creates an exhaustive list of all elements in $A \cup B$.

   **Answer:** The union is countable.

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