The Archimedean property; density of the rationals

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Practice problems

1. Prove that for any real number $a > 0$, there exists a natural number n such that $na > 1$.
2. Let x and y be real numbers such that $x < y$. Prove that there exists a rational number q such that $x < q < y$.
3. Prove that for any $\epsilon > 0$, there exists a rational number q such that $0 < q < \epsilon$.

Solutions

1. Prove that for any real number $a > 0$, there exists a natural number n such that $na > 1$.

   - Since $a > 0$, the number $1/a$ is a well-defined real number.
   - By the Archimedean property, for any real number x, there exists $n \in \mathbb{N}$ such that $n > x$.
   - Let $x = 1/a$. Then there exists $n \in \mathbb{N}$ such that $n > 1/a$.
   - Since $a > 0$, we can multiply both sides of the inequality by a without changing the inequality sign.
   - This gives $na > a(1/a)$, which simplifies to $na > 1$.

   Answer: The existence of n is guaranteed by the Archimedean property applied to $1/a$.

2. Let x and y be real numbers such that $x < y$. Prove that there exists a rational number q such that $x < q < y$.

   - Let $\delta = y - x$. Since $x < y$, $\delta > 0$.
   - By the Archimedean property, there exists $n \in \mathbb{N}$ such that $1/n < \delta$.
   - Consider the set $S = \{k \in \mathbb{Z} : k/n > x\}$. This set is non-empty because for large k, k/n exceeds x.
   - Let m be the smallest integer in S. Then $m/n > x$ and $(m-1)/n \leq x$.
   - From $(m-1)/n \leq x$, we have $m/n \leq x + 1/n$.
   - Since $1/n < y - x$, we have $x + 1/n < x + (y - x) = y$.
   - Thus, $x < m/n < y$, and $q = m/n$ is the required rational.

   Answer: The rational $q = m/n$ exists where $n > 1/(y - x)$ and $m = \lceil nx \rceil$ (if nx is not an integer).

3. Prove that for any $\epsilon > 0$, there exists a rational number q such that $0 < q < \epsilon$.

   - We are looking for a rational q in the interval $(0, \epsilon)$.
   - By the density of the rationals, for any two real numbers $x < y$, there exists a rational q such that $x < q < y$.
   - Let $x = 0$ and $y = \epsilon$.
   - Since $\epsilon > 0$, the condition $x < y$ is satisfied.
   - Therefore, there exists a rational q such that $0 < q < \epsilon$.

   Answer: Such a q exists by the density of the rationals in the interval $(0, \epsilon)$.
