Working with sup and inf — technique clinic

Lesson 14 · Module 1 — The Real Numbers · Real Analysis · dailymaths.uk/analysis

Practice problems

1. Let $S = \{1 - \frac{1}{n} : n \in \mathbb{N}\}$. Prove that $\sup S = 1$.
2. Let $S = \{\frac{n+1}{n} : n \in \mathbb{N}\}$. Find $\inf S$ and prove it using the epsilon characterisation.
3. Let $S = \{x \in \mathbb{R} : x^2 < 9\}$. Prove that $\sup S = 3$ and $\inf S = -3$.

Solutions

1. Let $S = \{1 - \frac{1}{n} : n \in \mathbb{N}\}$. Prove that $\sup S = 1$.

    - Step 1: Show 1 is an upper bound. Since $n \geq 1$, $\frac{1}{n} > 0$, so $1 - \frac{1}{n} < 1$ for all n.
    - Step 2: Use the epsilon characterisation. We want $1 - \frac{1}{n} > 1 - \epsilon$.
    - Step 3: Simplify the inequality: $-\frac{1}{n} > -\epsilon$, which means $\frac{1}{n} < \epsilon$, or $n > \frac{1}{\epsilon}$.
    - Step 4: Since the natural numbers are unbounded, we can always find an $n \in \mathbb{N}$ such that $n > \frac{1}{\epsilon}$. Thus, $\sup S = 1$.

    Answer: \sup S = 1

2. Let $S = \{\frac{n+1}{n} : n \in \mathbb{N}\}$. Find $\inf S$ and prove it using the epsilon characterisation.

    - Step 1: Guess $\inf S = 1$ because $\frac{n+1}{n} = 1 + \frac{1}{n}$, which approaches 1 as n grows.
    - Step 2: Show 1 is a lower bound. Since $\frac{1}{n} > 0$, $1 + \frac{1}{n} > 1$ for all n.
    - Step 3: We want $1 + \frac{1}{n} < 1 + \epsilon$. This simplifies to $\frac{1}{n} < \epsilon$, or $n > \frac{1}{\epsilon}$.
    - Step 4: Since the natural numbers are unbounded, we can pick $n > \frac{1}{\epsilon}$, ensuring an element of S is less than $1 + \epsilon$.

    Answer: \inf S = 1

3. Let $S = \{x \in \mathbb{R} : x^2 < 9\}$. Prove that $\sup S = 3$ and $\inf S = -3$.

    - Step 1: $x^2 < 9$ implies $-3 < x < 3$. Thus 3 is an upper bound and -3 is a lower bound.
    - Step 2: For $\sup S$, given $\epsilon > 0$, we need $x \in S$ such that $x > 3 - \epsilon$.
    - Step 3: If $\epsilon > 6$, $x = 0$ works. If $\epsilon \leq 6$, let $x = 3 - \frac{\epsilon}{2}$. Then $x^2 = (3 - \frac{\epsilon}{2})^2 < 3^2 = 9$, so $x \in S$.
    - Step 4: Similarly for $\inf S$, let $x = -3 + \frac{\epsilon}{2}$. Then $x^2 < 9$ and $x < -3 + \epsilon$.

    Answer: \sup S = 3, \inf S = -3
