Completeness: the supremum axiom — the one new idea

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Practice problems

1. Find the supremum of the set $S = \{\frac{n}{n+1} : n \in \mathbb{N}\}$. Does the set have a maximum?
2. Let $S = \{x \in \mathbb{R} : x^2 < 9\}$. Find the supremum and the maximum of S.
3. Prove that if a set S has a maximum m, then $\sup S = m$.

Solutions

1. Find the supremum of the set $S = \{\frac{n}{n+1} : n \in \mathbb{N}\}$. Does the set have a maximum?

   - Step 1: List some terms: $1/2, 2/3, 3/4, \ldots$. All terms are less than 1, so 1 is an upper bound.
   - Step 2: Check if 1 is the least upper bound. For any $\epsilon > 0$, we need $n/(n+1) > 1 - \epsilon$.
   - Step 3: Solve for n: $n > (1-\epsilon)(n+1) \implies n > n + 1 - \epsilon n - \epsilon \implies \epsilon n > 1 - \epsilon \implies n > (1-\epsilon)/\epsilon$.
   - Step 4: Since such an n always exists in the natural numbers, 1 is the supremum.
   - Step 5: Check if $1 \in S$. $n/(n+1) = 1$ would imply $n = n + 1$, which is $0 = 1$, impossible. So no maximum exists.

   Answer: \sup S = 1, \text no maximum exists

2. Let $S = \{x \in \mathbb{R} : x^2 < 9\}$. Find the supremum and the maximum of S.

   - Step 1: The condition $x^2 < 9$ is equivalent to $-3 < x < 3$.
   - Step 2: The set is the open interval $(-3, 3)$.
   - Step 3: Any number $M \geq 3$ is an upper bound. The smallest such number is 3.
   - Step 4: Since $3 \notin S$, there is no maximum.

   Answer: \sup S = 3, \text no maximum exists

3. Prove that if a set S has a maximum m, then $\sup S = m$.

   - Step 1: By definition of maximum, $m \in S$ and for all $x \in S, x \leq m$.
   - Step 2: This means m is an upper bound of S.
   - Step 3: To show it is the least upper bound, assume there is an upper bound $L < m$.
   - Step 4: But we know $m \in S$, and since L is an upper bound, $m \leq L$.
   - Step 5: This contradicts $L < m$. Therefore, no upper bound can be smaller than m.
   - Step 6: Thus, m is the least upper bound, so $\sup S = m$.

   Answer: \sup S = m
