Fields and order axioms — the rules we've been assuming

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Practice problems

1. Using the field and order axioms, prove that for any $x \in \mathbb{R}$, $x^2 \geq 0$.
2. Prove that if $a < b$ and $c < d$, then $a + c < b + d$.
3. Prove that if $0 < a < 1$, then $a^2 < a$.

Solutions

1. Using the field and order axioms, prove that for any $x \in \mathbb{R}$, $x^2 \geq 0$.

    - By the law of trichotomy, either $x > 0$, $x < 0$, or $x = 0$.
    - Case 1: If $x > 0$, then multiplying both sides of $x > 0$ by the positive number x gives $x^2 > 0$.
    - Case 2: If $x < 0$, then $-x > 0$. Multiplying both sides of $-x > 0$ by the positive number $-x$ gives $(-x)^2 > 0$, which simplifies to $x^2 > 0$.
    - Case 3: If $x = 0$, then $x^2 = 0 \cdot 0 = 0$.
    - In all cases, $x^2 \geq 0$.

    Answer: xˆ2 \ge 0

2. Prove that if $a < b$ and $c < d$, then $a + c < b + d$.

    - We are given $a < b$, which means $b - a > 0$.
    - We are given $c < d$, which means $d - c > 0$.
    - The sum of two positive numbers is positive, so $(b - a) + (d - c) > 0$.
    - Rearranging the terms using commutativity and associativity gives $(b + d) - (a + c) > 0$.
    - By the definition of order, this means $a + c < b + d$.

    Answer: a + c ¡ b + d

3. Prove that if $0 < a < 1$, then $a^2 < a$.

    - We are given $a < 1$.
    - We are also given $a > 0$.
    - Since a is positive, we can multiply both sides of the inequality $a < 1$ by a without changing the direction of the inequality.
    - Multiplying $a < 1$ by a yields $a \cdot a < 1 \cdot a$.
    - This simplifies to $a^2 < a$.

    Answer: aˆ2 ¡ a
