What's wrong with the rationals — the hole at root 2

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Practice problems

1. Starting with $x_1 = 1$, compute the fifth term x_5 of the Babylonian sequence $x_{n+1} = \frac{1}{2}(x_n + \frac{2}{x_n})$. Express your answer as a fraction.
2. Consider the sequence y_n formed by the first n digits of the decimal expansion of $\sqrt{2}$ (e.g., $y_1 = 1.4, y_2 = 1.41, \ldots$). Explain why this sequence consists only of rational numbers and why its limit is not rational.
3. If a set of numbers is dense, does that necessarily mean it is complete? Use the results of today's lesson to justify your answer.

Solutions

1. Starting with $x_1 = 1$, compute the fifth term x_5 of the Babylonian sequence $x_{n+1} = \frac{1}{2}(x_n + \frac{2}{x_n})$. Express your answer as a fraction.

    - We know $x_4 = \frac{577}{408}$.
    - Compute $\frac{2}{x_4} = \frac{2}{577/408} = \frac{816}{577}$.
    - Find the average: $x_5 = \frac{1}{2}(\frac{577}{408} + \frac{816}{577})$.
    - Common denominator is $408 \times 577 = 235416$.
    - Numerator is $577^2 + 816 \times 408 = 332929 + 332928 = 665857$.
    - Result is $\frac{665857}{2 \times 235416} = \frac{665857}{470832}$.

    Answer: $\frac{665857}{470832}$

2. Consider the sequence y_n formed by the first n digits of the decimal expansion of $\sqrt{2}$ (e.g., $y_1 = 1.4, y_2 = 1.41, \ldots$). Explain why this sequence consists only of rational numbers and why its limit is not rational.

    - Each y_n is a terminating decimal, which can be written as a fraction with a power of 10 in the denominator, so $y_n \in \mathbb{Q}$.
    - The sequence converges to $\sqrt{2}$ by construction.
    - Since $\sqrt{2}$ is irrational, the limit is not in $\mathbb{Q}$.

    Answer: The terms are terminating decimals (fractions), but the limit is $\sqrt{2}$, which is irrational.

3. If a set of numbers is dense, does that necessarily mean it is complete? Use the results of today's lesson to justify your answer.

    - No, density does not imply completeness.
    - The rational numbers $\mathbb{Q}$ are dense because between any two rationals there is another rational.
    - However, $\mathbb{Q}$ is not complete because the Babylonian sequence of rationals bunches up but its limit $\sqrt{2}$ is not in $\mathbb{Q}$.

    Answer: No; the rationals are dense but incomplete.
