Module 0 review + proof-writing self-test

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Practice problems

1. Prove that for any two real numbers x and y, if $xy = 0$, then $x = 0$ or $y = 0$.
2. Determine if the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^3$ is a bijection.
3. Prove by induction that for all $n \geq 1$, $1 + 2 + \cdots + n = \frac{n(n+1)}{2}$.

Solutions

1. Prove that for any two real numbers x and y, if $xy = 0$, then $x = 0$ or $y = 0$.
   - Use the contrapositive: assume it is not the case that ($x = 0$ or $y = 0$).
   - This means $x \neq 0$ and $y \neq 0$.
   - Since both are non-zero, their product xy cannot be zero.
   - Thus, $\neg(x = 0 \vee y = 0) \implies xy \neq 0$, which proves the original statement.

   Answer: Proven via contrapositive.

2. Determine if the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^3$ is a bijection.
   - Check injectivity: $x^3 = y^3$. Taking the cube root of both sides gives $x = y$. It is injective.
   - Check surjectivity: For any $y \in \mathbb{R}$, we can choose $x = \sqrt[3]{y}$. Then $f(x) = (\sqrt[3]{y})^3 = y$. It is surjective.
   - Since it is both injective and surjective, it is a bijection.

   Answer: Yes, it is a bijection.

3. Prove by induction that for all $n \geq 1$, $1 + 2 + \cdots + n = \frac{n(n+1)}{2}$.
   - Base case $n = 1$: $1 = \frac{1(2)}{2} = 1$. True.
   - Inductive step: Assume $S_k = \frac{k(k+1)}{2}$.
   - Then $S_{k+1} = S_k + (k+1) = \frac{k(k+1)}{2} + (k+1)$.
   - Factor out $(k+1)$: $(k+1)(\frac{k}{2} + 1) = (k+1)(\frac{k+2}{2}) = \frac{(k+1)(k+2)}{2}$.
   - This matches the formula for $n = k+1$.

   Answer: Proven via induction.
