Sets, functions, injections/surjections — the vocabulary

Lesson 9 · Module 0 — The Language of Proof · Real Analysis · dailymaths.uk/analysis

Practice problems

1. Determine if the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^3 + x$ is a bijection.
2. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = e^x$. Find the preimage of the set $T = [1, e^2]$.
3. Consider the function $f : \mathbb{Z} \to \mathbb{Z}$ defined by $f(n) = 2n$. Is this function injective? Is it surjective?

Solutions

1. Determine if the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^3 + x$ is a bijection.
   - To check injectivity, assume $f(x) = f(y)$, so $x^3 + x = y^3 + y$.
   - Rearrange to $x^3 - y^3 + x - y = 0$, which factors as $(x - y)(x^2 + xy + y^2 + 1) = 0$.
   - The term $x^2 + xy + y^2 + 1$ is always positive because it is $(x + y/2)^2 + 3y^2/4 + 1$.
   - Thus, we must have $x - y = 0$, so $x = y$. The function is injective.
   - For surjectivity, since $f(x)$ is a polynomial of odd degree, its limits as x goes to plus or minus infinity are plus and minus infinity.
   - By the Intermediate Value Theorem (Calculus, Lesson 20), $f(x)$ must take every real value. Thus, it is surjective.
   - Since it is both injective and surjective, it is a bijection.

   Answer: Yes, it is a bijection.

2. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = e^x$. Find the preimage of the set $T = [1, e^2]$.
   - The preimage $f^{-1}(T)$ is the set of all x such that $f(x) \in [1, e^2]$.
   - This means $1 \leq e^x \leq e^2$.
   - Taking the natural logarithm of all parts of the inequality, we get $\ln(1) \leq x \leq \ln(e^2)$.
   - Since $\ln(1) = 0$ and $\ln(e^2) = 2$, we have $0 \leq x \leq 2$.

   Answer: [0, 2]

3. Consider the function $f : \mathbb{Z} \to \mathbb{Z}$ defined by $f(n) = 2n$. Is this function injective? Is it surjective?
   - For injectivity, assume $f(n) = f(m)$, so $2n = 2m$. Dividing by 2 gives $n = m$. Thus, it is injective.
   - For surjectivity, we check if every integer k in the codomain $\mathbb{Z}$ has a preimage $n \in \mathbb{Z}$ such that $2n = k$.
   - If we pick $k = 1$, we need $2n = 1$, which means $n = 1/2$.
   - However, $1/2$ is not an integer, so $1/2 \notin \mathbb{Z}$.
   - Since there is no integer n such that $f(n) = 1$, the function is not surjective.

   Answer: Injective, but not surjective.
