Induction II — strong induction; Bernoulli's inequality

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Practice problems

1. Use mathematical induction to prove that for all $n \geq 1$, $\sum_{i=0}^{n} 2^i = 2^{n+1} - 1$.
2. Prove using strong induction that every integer $n \geq 2$ can be written as a product of primes.
3. Prove that for $x > 0$ and $n \geq 2$, $(1 + x)^n > 1 + nx$.

Solutions

1. Use mathematical induction to prove that for all $n \geq 1$, $\sum_{i=0}^{n} 2^i = 2^{n+1} - 1$.

   - Base case $n = 1$: $2^0 + 2^1 = 1 + 2 = 3$, and $2^{1+1} - 1 = 4 - 1 = 3$. The base case holds.
   - Inductive step: Assume $\sum_{i=0}^{k} 2^i = 2^{k+1} - 1$.
   - For $n = k + 1$, the sum is $(\sum_{i=0}^{k} 2^i) + 2^{k+1}$.
   - Substitute the hypothesis: $(2^{k+1} - 1) + 2^{k+1} = 2 \cdot 2^{k+1} - 1 = 2^{k+2} - 1$.
   - This matches the formula for $n = k + 1$.

   Answer: $\sum_{i=0}^{n} 2^i = 2^{n+1} - 1$

2. Prove using strong induction that every integer $n \geq 2$ can be written as a product of primes.

   - Base case $n = 2$: 2 is prime, so it is a product of one prime.
   - Inductive step: Assume every integer m such that $2 \leq m \leq k$ is a product of primes.
   - Consider $n = k + 1$. If $k + 1$ is prime, we are done.
   - If $k + 1$ is composite, $k + 1 = a \cdot b$ for some $1 < a, b < k + 1$.
   - By the strong inductive hypothesis, both a and b are products of primes.
   - Thus, $k + 1$ is a product of the primes that make up a and b.

   Answer: Every integer $n \geq 2$ is a product of primes.

3. Prove that for $x > 0$ and $n \geq 2$, $(1 + x)^n > 1 + nx$.

   - Base case $n = 2$: $(1 + x)^2 = 1 + 2x + x^2$. Since $x > 0$, $x^2 > 0$, so $1 + 2x + x^2 > 1 + 2x$.
   - Inductive step: Assume $(1 + x)^k > 1 + kx$ for some $k \geq 2$.
   - Multiply both sides by $(1 + x)$. Since $x > 0$, $(1 + x) > 0$.
   - $(1 + x)^{k+1} > (1 + kx)(1 + x) = 1 + x + kx + kx^2 = 1 + (k + 1)x + kx^2$.
   - Since $k \geq 2$ and $x^2 > 0$, the term kx^2 is strictly positive.
   - Therefore, $1 + (k + 1)x + kx^2 > 1 + (k + 1)x$.

   Answer: $(1 + x)^n > 1 + nx$
