Induction I — the principle and first proofs

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Practice problems

1. Prove by induction that for all $n \geq 1$, the sum of the first n squares is $1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$.
2. Prove by induction that $2^n > n$ for all positive integers n.
3. Prove by induction that $3^n > n$ for all positive integers n.

Solutions

1. Prove by induction that for all $n \geq 1$, the sum of the first n squares is $1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$.

   - Base case: For $n = 1$, $1^2 = 1$ and $\frac{1(2)(3)}{6} = 1$. True.
   - Inductive hypothesis: Assume $1^2 + 2^2 + \cdots + k^2 = \frac{k(k+1)(2k+1)}{6}$.
   - Inductive step: Add $(k+1)^2$ to both sides: $\frac{k(k+1)(2k+1)}{6} + (k+1)^2$.
   - Factor out $(k+1)$: $\frac{k+1}{6}[k(2k+1) + 6(k+1)] = \frac{k+1}{6}[2k^2 + 7k + 6]$.
   - Factor the quadratic: $2k^2 + 7k + 6 = (k+2)(2k+3)$.
   - Result: $\frac{(k+1)(k+2)(2k+3)}{6}$, which is the formula for $n = k+1$.

   Answer: \fracn(n+1)(2n+1)6

2. Prove by induction that $2^n > n$ for all positive integers n.

   - Base case: For $n = 1$, $2^1 = 2$ and $2 > 1$. True.
   - Inductive hypothesis: Assume $2^k > k$.
   - Inductive step: We want to show $2^{k+1} > k + 1$.
   - Start with $2^{k+1} = 2 \cdot 2^k$.
   - By hypothesis, $2 \cdot 2^k > 2k$.
   - Since $k \geq 1$, we know $2k = k + k \geq k + 1$.
   - Thus $2^{k+1} > k + 1$.

   Answer: 2ˆn ¿ n

3. Prove by induction that $3^n > n$ for all positive integers n.

   - Base case: For $n = 1$, $3^1 = 3$ and $3 > 1$. True.
   - Inductive hypothesis: Assume $3^k > k$ for some $k \geq 1$.
   - Inductive step: Consider $3^{k+1} = 3 \cdot 3^k$.
   - By hypothesis, $3 \cdot 3^k > 3k$.
   - We want to show $3k > k + 1$.
   - This is equivalent to $2k > 1$, which is true for all $k \geq 1$.
   - Thus $3^{k+1} > k + 1$.

   Answer: 3ˆn ¿ n
