Direct proof; proving implications

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Practice problems

1. Prove that the product of an even integer n and an odd integer m is even.
2. Prove that if n is an odd integer, then $n^2 - 1$ is even.
3. Prove that the sum of any three consecutive integers is always divisible by 3.

Solutions

1. Prove that the product of an even integer n and an odd integer m is even.

   - Assume n is even, so $n = 2k$ for some integer k.
   - Assume m is odd, so $m = 2j + 1$ for some integer j.
   - Compute the product: $nm = (2k)(2j + 1)$.
   - Distribute: $nm = 4kj + 2k$.
   - Factor out a 2: $nm = 2(2kj + k)$.
   - Since $2kj + k$ is an integer, nm is even by definition.

   Answer: The product is $2(2kj + k)$, which fits the definition of an even number.

2. Prove that if n is an odd integer, then $n^2 - 1$ is even.

   - Assume n is odd, so $n = 2k + 1$ for some integer k.
   - Substitute into the expression: $n^2 - 1 = (2k + 1)^2 - 1$.
   - Expand the square: $n^2 - 1 = (4k^2 + 4k + 1) - 1$.
   - Simplify: $n^2 - 1 = 4k^2 + 4k$.
   - Factor out a 2: $n^2 - 1 = 2(2k^2 + 2k)$.
   - Since $2k^2 + 2k$ is an integer, $n^2 - 1$ is even.

   Answer: The expression simplifies to $2(2k^2 + 2k)$, which is even.

3. Prove that the sum of any three consecutive integers is always divisible by 3.

   - Let the three consecutive integers be n, $n + 1$, and $n + 2$.
   - Sum them: $S = n + (n + 1) + (n + 2)$.
   - Combine like terms: $S = 3n + 3$.
   - Factor out the 3: $S = 3(n + 1)$.
   - Since $n + 1$ is an integer, S is a multiple of 3, and thus divisible by 3.

   Answer: The sum is $3(n + 1)$, which is divisible by 3.
