Negating statements — the skill behind every counterexample

Lesson 3 · Module 0 — The Language of Proof · Real Analysis · dailymaths.uk/analysis

Practice problems

1. Negate the following statement: $\exists L, \forall \epsilon > 0, \exists \delta > 0, \forall x, (|x - a| < \delta \implies |f(x) - L| < \epsilon)$
2. Negate the statement: For every real number x, there exists a real number y such that $x^2 + y^2 = 1$.
3. Negate the statement: There exists a function f such that for all x, y in the domain, if $x < y$, then $f(x) < f(y)$.

Solutions

1. Negate the following statement: $\exists L, \forall \epsilon > 0, \exists \delta > 0, \forall x, (|x-a| < \delta \implies |f(x) - L| < \epsilon)$

    - Flip the first quantifier: $\exists L$ becomes $\forall L$.
    - Flip the second quantifier: $\forall \epsilon > 0$ becomes $\exists \epsilon > 0$.
    - Flip the third quantifier: $\exists \delta > 0$ becomes $\forall \delta > 0$.
    - Flip the fourth quantifier: $\forall x$ becomes $\exists x$.
    - Negate the implication: $|x-a| < \delta \implies |f(x) - L| < \epsilon$ becomes $|x-a| < \delta$ and $|f(x) - L| \geq \epsilon$.

    Answer: $\forall L, \exists \epsilon > 0, \forall \delta > 0, \exists x, (|x-a| < \delta$ and $|f(x) - L| \geq \epsilon)$

2. Negate the statement: For every real number x, there exists a real number y such that $x^2+y^2 = 1$.

    - The original statement is $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}, x^2 + y^2 = 1$.
    - Flip $\forall x$ to $\exists x$.
    - Flip $\exists y$ to $\forall y$.
    - Negate the equation $x^2 + y^2 = 1$ to $x^2 + y^2 \neq 1$.

    Answer: There exists a real number x such that for all real numbers y, $x^2 + y^2 \neq 1$.

3. Negate the statement: There exists a function f such that for all x, y in the domain, if $x < y$, then $f(x) < f(y)$.

    - The original statement is $\exists f, \forall x, \forall y, (x < y \implies f(x) < f(y))$.
    - Flip $\exists f$ to $\forall f$.
    - Flip $\forall x$ to $\exists x$.
    - Flip $\forall y$ to $\exists y$.
    - Negate the implication: $x < y \implies f(x) < f(y)$ becomes $x < y$ and $f(x) \geq f(y)$.

    Answer: For all functions f, there exist x, y in the domain such that $x < y$ and $f(x) \geq f(y)$.
