Statements, quantifiers, and the order of forall and exists

Practice problems

1. Determine if the following statement is true or false: $\forall x \in \mathbb{R}, \exists y \in \mathbb{R} : xy = 1$.
2. Translate the following sentence into logical symbols: 'There is a real number that is greater than every other real number.'
3. Compare the truth values of (A) $\forall x \in \mathbb{R}, \exists y \in \mathbb{R} : x + y = 5$ and (B) $\exists y \in \mathbb{R}, \forall x \in \mathbb{R} : x + y = 5$.

Solutions

1. Determine if the following statement is true or false: $\forall x \in \mathbb{R}, \exists y \in \mathbb{R} : xy = 1$.

   - We start with an arbitrary x.
   - If $x \neq 0$, we can choose $y = 1/x$, then $x(1/x) = 1$.
   - However, if $x = 0$, we need to find y such that $0 \cdot y = 1$.
   - Since $0 \cdot y = 0$ for all y, no such y exists for $x = 0$.

   Answer: False

2. Translate the following sentence into logical symbols: 'There is a real number that is greater than every other real number.'

   - The phrase 'There is a real number' indicates an existential quantifier: $\exists y \in \mathbb{R}$.
   - The phrase 'greater than every other real number' indicates a universal quantifier for the other numbers: $\forall x \in \mathbb{R}$.
   - The condition is $y > x$ (though strictly 'every other' implies $x \neq y$, in analysis we usually write $y > x$ for all x to check for a maximum).

   Answer: $\exists y \in \mathbb{R}, \forall x \in \mathbb{R} : y > x$

3. Compare the truth values of (A) $\forall x \in \mathbb{R}, \exists y \in \mathbb{R} : x + y = 5$ and (B) $\exists y \in \mathbb{R}, \forall x \in \mathbb{R} : x + y = 5$.

   - For (A), given x, we can choose $y = 5 - x$. Then $x + (5 - x) = 5$. This works for all x, so (A) is true.
   - For (B), we must pick one y first. If we pick $y = 2$, then $x + 2 = 5$ only when $x = 3$. It fails for $x = 4$.
   - No single y can make $x + y = 5$ for every possible x.

   Answer: (A) is True, (B) is False
