# Why calculus needs a rebuild

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## Practice problems

1. Consider the function $f(x) = x$ if $x$ is rational and $f(x) = -x$ if $x$ is irrational. Is this function continuous at $x = 0$?
2. If a function $f$ is continuous on $[0, 1]$, must it be differentiable at $x = 1/2$? Provide a reason or a counterexample.
3. Suppose a sequence of functions $f_n(x)$ converges to $f(x)$ for all $x$ in $[0, 1]$. If each $f_n$ is a polynomial, must $f$ be a polynomial?

## Solutions

1. Consider the function $f(x) = x$ if $x$ is rational and $f(x) = -x$ if $x$ is irrational. Is this function continuous at $x = 0$?

   - To check continuity at $x = 0$, we need to see if $\lim_{x \to 0} f(x) = f(0)$.
   - Note that $f(0) = 0$ because 0 is rational.
   - For any $x$, the value of $f(x)$ is either $x$ or $-x$.
   - In both cases, $|f(x) - 0| = |x|$.
   - As $x$ approaches 0, $|x|$ approaches 0 regardless of whether $x$ is rational or irrational.
   - Therefore, the limit is 0, which equals $f(0)$.

   **Answer:** Yes, it is continuous at $x = 0$.

2. If a function $f$ is continuous on $[0, 1]$, must it be differentiable at $x = 1/2$? Provide a reason or a counterexample.

   - Recall the definition of differentiability: the limit of the difference quotient must exist.
   - Consider the function $f(x) = |x - 1/2|$.
   - This function is continuous everywhere on $[0, 1]$ because it is the composition of continuous functions.
   - However, at $x = 1/2$, the function has a sharp corner.
   - The left-hand derivative is $-1$ and the right-hand derivative is 1.
   - Since the one-sided limits of the difference quotient are not equal, the derivative does not exist.

   **Answer:** No, the function $f(x) = |x - 1/2|$ is a counterexample.

3. Suppose a sequence of functions $f_n(x)$ converges to $f(x)$ for all $x$ in $[0, 1]$. If each $f_n$ is a polynomial, must $f$ be a polynomial?

   - Polynomials are smooth and have a finite number of terms.
   - Consider the Taylor series for the exponential function $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$.
   - Let $f_n(x)$ be the $n$-th partial sum of this series, which is a polynomial of degree $n$.
   - As $n \to \infty$, $f_n(x)$ converges to $f(x) = e^x$ for all $x$.
   - While each $f_n$ is a polynomial, the limit function $e^x$ is not a polynomial (it has infinitely many non-zero derivatives).

   **Answer:** No, the limit of polynomials can be a non-polynomial function like $e^x$.

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