

Grand review & epilogue

Lesson 60 · Module 5 — Applications & Epilogue · Multivariable & Vector Calculus · daily-maths.uk/vectorcalc

Practice problems

1. Compute the total flux of the vector field $\mathbf{F} = \langle x, y, z \rangle$ through the surface of a cube centered at the origin with side length 2.
2. Use Green's Theorem to find the area of the region enclosed by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.
3. A vector field is given by $\mathbf{F} = \langle 2xy, x^2 + z^2, 2yz \rangle$. Determine if the field is conservative. If so, find the potential function ϕ .

Solutions

1. Compute the total flux of the vector field $\mathbf{F} = \langle x, y, z \rangle$ through the surface of a cube centered at the origin with side length 2.

- Step 1: Compute the divergence of $\mathbf{F}$. $\nabla \cdot \mathbf{F} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) = 1 + 1 + 1 = 3$.
- Step 2: Use the Divergence Theorem. The flux is the triple integral of the divergence over the volume of the cube.
- Step 3: The volume of a cube with side length 2 is $2^3 = 8$.
- Step 4: The flux is $\iiint_V 3dV = 3 \times 8 = 24$.

Answer: 24

2. Use Green's Theorem to find the area of the region enclosed by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

- Step 1: Use the area formula $\text{Area} = \oint_C \frac{1}{2}(xdy - ydx)$.
- Step 2: Parameterize the ellipse: $x = a \cos t$, $y = b \sin t$ for $t \in [0, 2\pi]$.
- Step 3: Compute differentials: $dx = -a \sin t dt$, $dy = b \cos t dt$.
- Step 4: Substitute into the integral: $\int_0^{2\pi} \frac{1}{2}((a \cos t)(b \cos t) - (b \sin t)(-a \sin t))dt$.
- Step 5: Simplify: $\int_0^{2\pi} \frac{1}{2}ab(\cos^2 t + \sin^2 t)dt = \int_0^{2\pi} \frac{1}{2}abdt$.
- Step 6: Evaluate: $\frac{1}{2}ab(2\pi) = \pi ab$.

Answer: πab

3. A vector field is given by $\mathbf{F} = \langle 2xy, x^2 + z^2, 2yz \rangle$. Determine if the field is conservative. If so, find the potential function ϕ .

- Step 1: Check the curl. $\nabla \times \mathbf{F} = \langle \frac{\partial}{\partial y}(2yz) - \frac{\partial}{\partial z}(x^2 + z^2), \frac{\partial}{\partial z}(2xy) - \frac{\partial}{\partial x}(2yz), \frac{\partial}{\partial x}(x^2 + z^2) - \frac{\partial}{\partial y}(2xy) \rangle$.
- Step 2: Compute components: $\langle 2z - 2z, 0 - 0, 2x - 2x \rangle = \langle 0, 0, 0 \rangle$. The field is conservative.
- Step 3: Integrate F_x with respect to x : $\phi = \int 2xy dx = x^2 y + g(y, z)$.
- Step 4: Differentiate ϕ with respect to y : $\frac{\partial \phi}{\partial y} = x^2 + \frac{\partial g}{\partial y}$. Match with $F_y = x^2 + z^2$, so $\frac{\partial g}{\partial y} = z^2$.
- Step 5: Integrate $\frac{\partial g}{\partial y}$ with respect to y : $g(y, z) = yz^2 + h(z)$. Thus $\phi = x^2 y + yz^2 + h(z)$.
- Step 6: Differentiate ϕ with respect to z : $\frac{\partial \phi}{\partial z} = 2yz + h'(z)$. Match with $F_z = 2yz$, so $h'(z) = 0$.
- Step 7: Final potential: $\phi(x, y, z) = x^2 y + yz^2 + C$.

Answer: $\phi(x, y, z) = x^2 y + yz^2 + C$