

A glimpse of differential forms

Lesson 59 · Module 5 — Applications & Epilogue · Multivariable & Vector Calculus · daily-maths.uk/vectorcalc

Practice problems

1. Given the vector field $\mathbf{F} = \langle 3x^2, 2y \rangle$, write the corresponding 1-form ω and compute its exterior derivative $d\omega$.
2. Consider the 1-form $\omega = y^2 dx + x^2 dy$. Compute $d\omega$ and explain what this result represents in terms of vector calculus.
3. If ω is a 2-form in 3D space given by $\omega = xz dy \wedge dz + yz dz \wedge dx + xy dx \wedge dy$, compute $d\omega$.

Solutions

1. Given the vector field $\mathbf{F} = \langle 3x^2, 2y \rangle$, write the corresponding 1-form ω and compute its exterior derivative $d\omega$.

- The 1-form is constructed by replacing the components of the vector field with coefficients of dx and dy : $\omega = 3x^2dx + 2ydy$.
- The exterior derivative is $d\omega = (\frac{\partial(2y)}{\partial x} - \frac{\partial(3x^2)}{\partial y})dx \wedge dy$.
- Computing the partials: $\frac{\partial(2y)}{\partial x} = 0$ and $\frac{\partial(3x^2)}{\partial y} = 0$.
- Thus, $d\omega = (0 - 0)dx \wedge dy = 0$.

Answer: $d\omega = 0$

2. Consider the 1-form $\omega = y^2dx + x^2dy$. Compute $d\omega$ and explain what this result represents in terms of vector calculus.

- The exterior derivative is $d\omega = (\frac{\partial(x^2)}{\partial x} - \frac{\partial(y^2)}{\partial y})dx \wedge dy$.
- Computing the partials: $\frac{\partial(x^2)}{\partial x} = 2x$ and $\frac{\partial(y^2)}{\partial y} = 2y$.
- The result is $d\omega = (2x - 2y)dx \wedge dy$.
- In vector calculus, this represents the 2D curl (or the k-component of the curl) of the vector field $\mathbf{F} = \langle y^2, x^2 \rangle$.

Answer: $d\omega = (2x - 2y) dx \wedge dy$

3. If ω is a 2-form in 3D space given by $\omega = xzdy \wedge dz + yzdz \wedge dx + xydx \wedge dy$, compute $d\omega$.

- The exterior derivative of a 2-form is $d\omega = (\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z})dx \wedge dy \wedge dz$, where P, Q, R are the coefficients of $dy \wedge dz, dz \wedge dx, dx \wedge dy$ respectively.
- Here $P = xz$, $Q = yz$, and $R = xy$.
- Compute partials: $\frac{\partial(xz)}{\partial x} = z$, $\frac{\partial(yz)}{\partial y} = z$, and $\frac{\partial(xy)}{\partial z} = 0$.
- Summing them: $z + z + 0 = 2z$.
- The result is $d\omega = 2zdx \wedge dy \wedge dz$.

Answer: $d\omega = 2z dx \wedge dy \wedge dz$