

Heat, waves, and Laplace revisited in vector language

Lesson 58 · Module 5 — Applications & Epilogue · Multivariable & Vector Calculus · daily-maths.uk/vectorcalc

Practice problems

1. Determine if the function $u(x, y, z) = x^2 - y^2 + z^2$ satisfies Laplace's equation $\nabla^2 u = 0$.
2. Given the heat equation $u_t = k\nabla^2 u$ with $k = 0.5$, find the rate of change of temperature u_t at the origin for the distribution $u(x, y) = 10 - (x^2 + 2y^2)$.
3. Verify if $u(x, t) = e^{-t} \sin(x)$ satisfies the heat equation $u_t = k\nabla^2 u$ for some constant k . If so, find k .

Solutions

1. Determine if the function $u(x, y, z) = x^2 - y^2 + z^2$ satisfies Laplace's equation $\nabla^2 u = 0$.

- Compute the gradient: $\nabla u = [2x, -2y, 2z]$.
- Compute the divergence of the gradient: $\nabla \cdot [2x, -2y, 2z] = \frac{\partial}{\partial x}(2x) + \frac{\partial}{\partial y}(-2y) + \frac{\partial}{\partial z}(2z)$.
- Simplify the derivatives: $2 - 2 + 2 = 2$.
- Since $2 \neq 0$, the function does not satisfy Laplace's equation.

Answer: No, $\nabla^2 u = 2$

2. Given the heat equation $u_t = k\nabla^2 u$ with $k = 0.5$, find the rate of change of temperature u_t at the origin for the distribution $u(x, y) = 10 - (x^2 + 2y^2)$.

- Find the first partials: $u_x = -2x$ and $u_y = -4y$.
- Find the second partials: $u_{xx} = -2$ and $u_{yy} = -4$.
- Compute the Laplacian: $\nabla^2 u = -2 - 4 = -6$.
- Apply the heat equation: $u_t = 0.5 \cdot (-6) = -3$.

Answer: -3

3. Verify if $u(x, t) = e^{-t} \sin(x)$ satisfies the heat equation $u_t = k\nabla^2 u$ for some constant k . If so, find k .

- Compute the time derivative: $u_t = -e^{-t} \sin(x)$.
- Compute the spatial second derivative: $u_x = e^{-t} \cos(x)$, so $u_{xx} = -e^{-t} \sin(x)$.
- Set up the equation: $-e^{-t} \sin(x) = k(-e^{-t} \sin(x))$.
- Divide both sides by $-e^{-t} \sin(x)$ to find $k = 1$.

Answer: Yes, $k = 1$