

The continuity equation — conservation laws from the divergence theorem

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Practice problems

1. Given a density $\rho(x, y, z, t) = e^{-3t}(x^2 + y^2 + z^2)$ and a current density $\mathbf{J} = e^{-3t}(2x, 2y, 2z)$, check if the continuity equation $\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0$ is satisfied.
2. A fluid has a constant density $\rho = 1$. Find the condition on the velocity field $\mathbf{v}$ such that the continuity equation is satisfied, given $\mathbf{J} = \rho \mathbf{v}$.
3. Suppose $\mathbf{J} = (t, 0, 0)$ and the continuity equation is satisfied. Find the general form of the density $\rho(x, y, z, t)$.

Solutions

1. Given a density $\rho(x, y, z, t) = e^{-3t}(x^2 + y^2 + z^2)$ and a current density $\mathbf{J} = e^{-3t}(2x, 2y, 2z)$, check if the continuity equation $\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0$ is satisfied.

- Compute the time derivative of density: $\frac{\partial \rho}{\partial t} = -3e^{-3t}(x^2 + y^2 + z^2)$.
- Compute the divergence of $\mathbf{J}$: $\nabla \cdot \mathbf{J} = e^{-3t}(2 + 2 + 2) = 6e^{-3t}$.
- Sum them: $-3e^{-3t}(x^2 + y^2 + z^2) + 6e^{-3t}$.
- Observe that this sum is not zero for all x, y, z .

Answer: Not satisfied.

2. A fluid has a constant density $\rho = 1$. Find the condition on the velocity field $\mathbf{v}$ such that the continuity equation is satisfied, given $\mathbf{J} = \rho \mathbf{v}$.

- Substitute $\rho = 1$ into the continuity equation: $\frac{\partial(1)}{\partial t} + \nabla \cdot (1\mathbf{v}) = 0$.
- The time derivative of a constant is 0.
- The equation simplifies to $\nabla \cdot \mathbf{v} = 0$.

Answer: The velocity field must be solenoidal (divergence-free), $\nabla \cdot \mathbf{v} = 0$.

3. Suppose $\mathbf{J} = (t, 0, 0)$ and the continuity equation is satisfied. Find the general form of the density $\rho(x, y, z, t)$.

- Compute the divergence of $\mathbf{J}$: $\nabla \cdot \mathbf{J} = \frac{\partial}{\partial x}(t) + 0 + 0 = 0$.
- The continuity equation becomes $\frac{\partial \rho}{\partial t} + 0 = 0$.
- This means ρ does not depend on t .
- Therefore, ρ can be any function of space only: $\rho = f(x, y, z)$.

Answer: $\rho = f(x, y, z)$