

Module 4 Review + Self-Test

Lesson 56 · Module 4 — Green, Stokes & Divergence · Multivariable & Vector Calculus · dailymaths.uk/vectorcalc

Practice problems

1. Compute the flux of $\mathbf{F} = \langle x^2, y^2, z^2 \rangle$ through the surface of the unit cube $V = [0, 1] \times [0, 1] \times [0, 1]$ using the Divergence Theorem.
2. Use Green's Theorem to evaluate $\oint_C (y^2 dx + 3xy dy)$ where C is the boundary of the semi-circle $x^2 + y^2 \leq 1, y \geq 0$ oriented counter-clockwise.
3. A vector field is given by $\mathbf{F} = \langle z, x, y \rangle$. Find the circulation around the triangle with vertices $(1, 0, 0), (0, 1, 0), (0, 0, 1)$ using Stokes' Theorem.

Solutions

1. Compute the flux of $\mathbf{F} = \langle x^2, y^2, z^2 \rangle$ through the surface of the unit cube $V = [0, 1] \times [0, 1] \times [0, 1]$ using the Divergence Theorem.
 - Compute the divergence: $\nabla \cdot \mathbf{F} = 2x + 2y + 2z$.
 - Set up the triple integral over the cube: $\int_0^1 \int_0^1 \int_0^1 (2x + 2y + 2z) dx dy dz$.
 - Integrate $2x$ over the cube: $\int_0^1 2x dx = 1$, and the other dimensions contribute 1×1 . Total for $2x$ is 1.
 - By symmetry, the integrals of $2y$ and $2z$ are also 1.
 - Sum the results: $1 + 1 + 1 = 3$.

Answer: 3

2. Use Green's Theorem to evaluate $\oint_C (y^2 dx + 3xy dy)$ where C is the boundary of the semi-circle $x^2 + y^2 \leq 1, y \geq 0$ oriented counter-clockwise.
 - Identify $P = y^2$ and $Q = 3xy$.
 - Compute the curl: $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 3y - 2y = y$.
 - Set up the double integral over the semi-disk: $\iint_D y dA$.
 - Use polar coordinates: $\int_0^\pi \int_0^1 (r \sin \theta) r dr d\theta$.
 - Integrate r^2 from 0 to 1 to get $1/3$.
 - Integrate $\sin \theta$ from 0 to π to get 2.
 - Multiply: $1/3 \times 2 = 2/3$.

Answer: 2/3

3. A vector field is given by $\mathbf{F} = \langle z, x, y \rangle$. Find the circulation around the triangle with vertices $(1, 0, 0), (0, 1, 0), (0, 0, 1)$ using Stokes' Theorem.
 - Compute the curl: $\nabla \times \mathbf{F} = \langle 1 - 0, 1 - 0, 1 - 0 \rangle = \langle 1, 1, 1 \rangle$.
 - The surface S is the plane $x + y + z = 1$. The unit normal vector is $\mathbf{n} = \frac{1}{\sqrt{3}} \langle 1, 1, 1 \rangle$.
 - Compute the dot product: $(\nabla \times \mathbf{F}) \cdot \mathbf{n} = \langle 1, 1, 1 \rangle \cdot \frac{1}{\sqrt{3}} \langle 1, 1, 1 \rangle = \frac{3}{\sqrt{3}} = \sqrt{3}$.
 - The area of the triangle is $\frac{1}{2} |\mathbf{AB} \times \mathbf{AC}|$. $\mathbf{AB} = \langle -1, 1, 0 \rangle, \mathbf{AC} = \langle -1, 0, 1 \rangle$.
 - Cross product is $\langle 1, 1, 1 \rangle$, magnitude is $\sqrt{3}$. Area is $\sqrt{3}/2$.
 - Integral is $\sqrt{3} \times \frac{\sqrt{3}}{2} = 3/2$.

Answer: 3/2