

Maxwell's equations — the vocabulary of light, finally readable

Lesson 55 · Module 4 — Green, Stokes & Divergence · Multivariable & Vector Calculus · dailymaths.uk/vectorcalc

Practice problems

1. Given a charge density $\rho(x, y, z) = 3x^2 + 2y^2 + z^2$ inside a unit cube $V = [0, 1] \times [0, 1] \times [0, 1]$, use the Divergence Theorem and Maxwell's first equation to find the total electric flux $\iint_S \mathbf{E} \cdot \mathbf{n} dS$ through the surface of the cube.
2. A magnetic field is given by $\mathbf{B} = (y^2, x^2, 0)$. Show that this field satisfies Maxwell's second equation and explain what this means physically.
3. A changing magnetic field creates an electric field such that $\nabla \times \mathbf{E} = (0, 0, -2t)$. Use Stokes' Theorem to find the voltage $\oint_C \mathbf{E} \cdot d\mathbf{r}$ around a circle of radius $R = 2$ in the xy -plane.

Solutions

1. Given a charge density $\rho(x, y, z) = 3x^2 + 2y^2 + z^2$ inside a unit cube $V = [0, 1] \times [0, 1] \times [0, 1]$, use the Divergence Theorem and Maxwell's first equation to find the total electric flux $\iint_S \mathbf{E} \cdot \mathbf{n} dS$ through the surface of the cube.

- By the Divergence Theorem, the flux is $\iiint_V (\nabla \cdot \mathbf{E}) dV$.
- Substitute Maxwell's first equation: $\iiint_V \frac{\rho}{\epsilon_0} dV$.
- Compute the integral: $\frac{1}{\epsilon_0} \int_0^1 \int_0^1 \int_0^1 (3x^2 + 2y^2 + z^2) dx dy dz$.
- The integral is $[x^3]_0^1 + [\frac{2}{3}y^3]_0^1 + [\frac{1}{3}z^3]_0^1 = 1 + \frac{2}{3} + \frac{1}{3} = 2$.
- The total flux is $2/\epsilon_0$.

Answer: $2/\epsilon_0$

2. A magnetic field is given by $\mathbf{B} = (y^2, x^2, 0)$. Show that this field satisfies Maxwell's second equation and explain what this means physically.

- Compute the divergence: $\nabla \cdot \mathbf{B} = \frac{\partial}{\partial x}(y^2) + \frac{\partial}{\partial y}(x^2) + \frac{\partial}{\partial z}(0)$.
- The result is $0 + 0 + 0 = 0$.
- Since the divergence is zero, it satisfies the equation $\nabla \cdot \mathbf{B} = 0$.
- Physically, this means there are no magnetic monopoles in this field; every field line that enters a region must leave it.

Answer: The divergence is 0, so it satisfies the equation; no monopoles are present.

3. A changing magnetic field creates an electric field such that $\nabla \times \mathbf{E} = (0, 0, -2t)$. Use Stokes' Theorem to find the voltage $\oint_C \mathbf{E} \cdot d\mathbf{r}$ around a circle of radius $R = 2$ in the xy -plane.

- By Stokes' Theorem, $\oint_C \mathbf{E} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{E}) \cdot \mathbf{n} dS$.
- The normal vector to the xy -plane is $\mathbf{n} = (0, 0, 1)$.
- The dot product is $(0, 0, -2t) \cdot (0, 0, 1) = -2t$.
- The integral is $\iint_S -2t dS = -2t \times \text{Area of circle}$.
- Area is $\pi R^2 = \pi(2^2) = 4\pi$.
- The voltage is $-2t(4\pi) = -8\pi t$.

Answer: $-8\pi t$