Stokes' theorem II — computations; consistency with Green

Lesson 53 · Module 4 — Green, Stokes & Divergence · Multivariable & Vector Calculus · dailymaths.uk/vectorcalc

Practice problems

1. Compute the circulation of $\mathbf{F} = [xy, xz, yz]$ around the triangle with vertices $(0,0,0)$, $(1,0,0)$, and $(0,1,0)$ using Stokes' Theorem.
2. Use Stokes' Theorem to find the circulation of $\mathbf{F} = [z^2, x^2, y^2]$ around the circle $x^2 + y^2 = 4$ in the plane $z = 3$.
3. Verify Stokes' Theorem for $\mathbf{F} = [-y, x, 0]$ around the boundary of the upper hemisphere $x^2 + y^2 + z^2 = 1, z \geq 0$.

Solutions

1. Compute the circulation of $\mathbf{F} = [xy, xz, yz]$ around the triangle with vertices $(0,0,0)$, $(1,0,0)$, and $(0,1,0)$ using Stokes' Theorem.

   - Compute the curl: $\nabla \times \mathbf{F} = [z - x, 0 - z, z - y] = [z - x, -z, z - y]$.
   - The surface is the triangle in the xy-plane, so $z = 0$ and $\mathbf{n} = [0, 0, 1]$.
   - The dot product is $(\nabla \times \mathbf{F}) \cdot \mathbf{n} = z - y$. Since $z = 0$ on the surface, the integrand is $-y$.
   - Integrate $-y$ over the triangle: $\int_0^1 \int_0^{1-x} -y dy dx = \int_0^1 [-\frac{1}{2}y^2]_0^{1-x} dx = \int_0^1 -\frac{1}{2}(1-x)^2 dx$.
   - Evaluating the integral: $[\frac{1}{6}(1-x)^3]_0^1 = 0 - \frac{1}{6} = -\frac{1}{6}$.

   Answer: -\frac16

2. Use Stokes' Theorem to find the circulation of $\mathbf{F} = [z^2, x^2, y^2]$ around the circle $x^2 + y^2 = 4$ in the plane $z = 3$.

   - Compute the curl: $\nabla \times \mathbf{F} = [2y - 0, 2z - 0, 2x - 0] = [2y, 2z, 2x]$.
   - The surface is a flat disk at $z = 3$ with $\mathbf{n} = [0, 0, 1]$.
   - The dot product is $(\nabla \times \mathbf{F}) \cdot \mathbf{n} = 2x$.
   - Integrate $2x$ over the disk $x^2 + y^2 \leq 4$.
   - By symmetry, the integral of x over a disk centered at the origin is zero.

   Answer: 0

3. Verify Stokes' Theorem for $\mathbf{F} = [-y, x, 0]$ around the boundary of the upper hemisphere $x^2 + y^2 + z^2 = 1, z \geq 0$.

   - Line integral: C is the unit circle in xy-plane. $\mathbf{r}(t) = [\cos t, \sin t, 0]$, $\mathbf{r}'(t) = [-\sin t, \cos t, 0]$. $\mathbf{F} \cdot \mathbf{r}' = [-\sin t, \cos t, 0] \cdot [-\sin t, \cos t, 0] = 1$. $\int_0^{2\pi} 1 dt = 2\pi$.
   - Surface integral: $\nabla \times \mathbf{F} = [0, 0, 2]$.
   - Using the flat disk S as the surface: $\mathbf{n} = [0, 0, 1]$. $\iint_S [0, 0, 2] \cdot [0, 0, 1] dS = 2 \times \text{Area(disk)} = 2\pi$.
   - Both sides equal 2π.

   Answer: 2\pi