

Stokes' Theorem I — Statement and Intuition

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Practice problems

1. Use Stokes' Theorem to compute the circulation of $\mathbf{F} = \langle zx, xy, yz \rangle$ around the boundary of the triangle with vertices $(0, 0, 0)$, $(2, 0, 0)$, and $(0, 2, 0)$.
2. Explain why the flux of the curl of any smooth vector field $\mathbf{F}$ through a closed sphere is always zero using Stokes' Theorem.
3. Calculate the circulation of $\mathbf{F} = \langle -y, x, e^{z^2} \rangle$ around the circle $x^2 + y^2 = 9$ in the plane $z = 2$.

Solutions

1. Use Stokes' Theorem to compute the circulation of $\mathbf{F} = \langle zx, xy, yz \rangle$ around the boundary of the triangle with vertices $(0, 0, 0)$, $(2, 0, 0)$, and $(0, 2, 0)$.

- Step 1: Compute the curl of $\mathbf{F}$. $\nabla \times \mathbf{F} = \langle z - x, x - y, y - z \rangle$.
- Step 2: Identify the surface S . The simplest surface is the flat triangle in the xy -plane where $z = 0$.
- Step 3: Determine the normal vector. For the xy -plane, $\mathbf{n} = \langle 0, 0, 1 \rangle$.
- Step 4: Compute the dot product $(\nabla \times \mathbf{F}) \cdot \mathbf{n} = y - z$. Since $z = 0$ on the surface, this is just y .
- Step 5: Integrate y over the triangle region D bounded by $x = 0, y = 0, x + y = 2$.
 $\int_0^2 \int_0^{2-x} y dy dx = \int_0^2 \frac{1}{2}(2-x)^2 dx = [-\frac{1}{6}(2-x)^3]_0^2 = 0 - (-\frac{8}{6}) = \frac{4}{3}$.

Answer: $4/3$

2. Explain why the flux of the curl of any smooth vector field $\mathbf{F}$ through a closed sphere is always zero using Stokes' Theorem.

- Step 1: A closed sphere has no boundary curve C .
- Step 2: According to Stokes' Theorem, the surface integral of the curl equals the line integral around the boundary.
- Step 3: Since the boundary is empty (or a single point), the line integral is zero.
- Step 4: Therefore, the surface integral of the curl over any closed surface must be zero.

Answer: The boundary curve is empty, so the line integral is zero.

3. Calculate the circulation of $\mathbf{F} = \langle -y, x, e^{z^2} \rangle$ around the circle $x^2 + y^2 = 9$ in the plane $z = 2$.

- Step 1: Compute the curl. $\nabla \times \mathbf{F} = \langle 0, 0, 1 - (-1) \rangle = \langle 0, 0, 2 \rangle$.
- Step 2: Choose the surface S as the flat disk $x^2 + y^2 \leq 9$ at $z = 2$.
- Step 3: The normal vector is $\mathbf{n} = \langle 0, 0, 1 \rangle$.
- Step 4: The dot product is $(\nabla \times \mathbf{F}) \cdot \mathbf{n} = 2$.
- Step 5: Integrate 2 over the area of the disk with radius 3. $\iint_S 2 dS = 2 \times (\pi \cdot 3^2) = 18\pi$.

Answer: 18π