The Divergence Theorem II — computations and Gauss's law

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Practice problems

1. Compute the flux of $\mathbf{F} = \langle x, 2y, 3z \rangle$ through the surface of a sphere of radius 2 centered at the origin.
2. Find the flux of $\mathbf{F} = \langle x^3, y^3, z^3 \rangle$ through the unit cube $[0,1] \times [0,1] \times [0,1]$.
3. A vector field is given by $\mathbf{F} = \langle x, y, e^z \rangle$. Calculate the flux through the closed surface bounding the cylinder $x^2 + y^2 \leq 1$ and $0 \leq z \leq 1$.

Solutions

1. Compute the flux of $\mathbf{F} = \langle x, 2y, 3z \rangle$ through the surface of a sphere of radius 2 centered at the origin.

   - Compute the divergence: $\nabla \cdot \mathbf{F} = 1 + 2 + 3 = 6$.
   - Apply the Divergence Theorem: $\Phi = \iiint_V 6 dV$.
   - The volume of a sphere is $\frac{4}{3}\pi r^3$. For $r = 2$, $V = \frac{4}{3}\pi(8) = \frac{32\pi}{3}$.
   - The flux is $6 \cdot \frac{32\pi}{3} = 64\pi$.

   Answer: 64\pi

2. Find the flux of $\mathbf{F} = \langle x^3, y^3, z^3 \rangle$ through the unit cube $[0,1] \times [0,1] \times [0,1]$.

   - Compute the divergence: $\nabla \cdot \mathbf{F} = 3x^2 + 3y^2 + 3z^2$.
   - Set up the triple integral: $\int_0^1 \int_0^1 \int_0^1 (3x^2 + 3y^2 + 3z^2) dx dy dz$.
   - Integrate each term: $\int_0^1 3x^2 dx = 1$. By symmetry, the other two terms are also 1.
   - The total flux is $1 + 1 + 1 = 3$.

   Answer: 3

3. A vector field is given by $\mathbf{F} = \langle x, y, e^z \rangle$. Calculate the flux through the closed surface bounding the cylinder $x^2 + y^2 \leq 1$ and $0 \leq z \leq 1$.

   - Compute the divergence: $\nabla \cdot \mathbf{F} = 1 + 1 + e^z = 2 + e^z$.
   - Use cylindrical coordinates: $\int_0^{2\pi} \int_0^1 \int_0^1 (2 + e^z) r dz dr d\theta$.
   - Integrate with respect to z: $[2z + e^z]_0^1 = 2 + e - 1 = 1 + e$.
   - Integrate with respect to r and θ: $(1 + e) \cdot \pi(1)^2 = \pi(1 + e)$.

   Answer: \pi(1 + e)