

Flux integrals — flow through a membrane

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Practice problems

1. Compute the flux of $\mathbf{F} = \langle z, x, y \rangle$ through the unit disk $x^2 + y^2 \leq 1$ in the xy -plane, oriented upwards.
2. Find the flux of $\mathbf{F} = \langle 0, 0, x^2 + y^2 \rangle$ through the cylinder $x^2 + y^2 = 1$ for $0 \leq z \leq 2$, oriented outwards.
3. Compute the flux of $\mathbf{F} = \langle x, y, 0 \rangle$ through the upper hemisphere $x^2 + y^2 + z^2 = 1, z \geq 0$, oriented upwards.

Solutions

1. Compute the flux of $\mathbf{F} = \langle z, x, y \rangle$ through the unit disk $x^2 + y^2 \leq 1$ in the xy -plane, oriented upwards.

- The surface is $z = 0$, so the parameterisation is $\mathbf{r}(u, v) = \langle u, v, 0 \rangle$ with $u^2 + v^2 \leq 1$.
- The normal vector is $\mathbf{r}_u \times \mathbf{r}_v = \langle 0, 0, 1 \rangle$.
- Substitute the surface into the field: $\mathbf{F}(\mathbf{r}(u, v)) = \langle 0, u, v \rangle$.
- Compute the dot product: $\langle 0, u, v \rangle \cdot \langle 0, 0, 1 \rangle = v$.
- Integrate v over the unit disk: $\iint_D v \, du \, dv$.
- By symmetry, the integral of v over a disk centered at the origin is zero.

Answer: 0

2. Find the flux of $\mathbf{F} = \langle 0, 0, x^2 + y^2 \rangle$ through the cylinder $x^2 + y^2 = 1$ for $0 \leq z \leq 2$, oriented outwards.

- Parameterise the cylinder: $\mathbf{r}(\theta, z) = \langle \cos \theta, \sin \theta, z \rangle$ for $0 \leq \theta \leq 2\pi, 0 \leq z \leq 2$.
- Compute tangents: $\mathbf{r}_\theta = \langle -\sin \theta, \cos \theta, 0 \rangle$ and $\mathbf{r}_z = \langle 0, 0, 1 \rangle$.
- Cross product: $\mathbf{r}_\theta \times \mathbf{r}_z = \langle \cos \theta, \sin \theta, 0 \rangle$. This points outwards.
- Substitute into field: $\mathbf{F} = \langle 0, 0, \cos^2 \theta + \sin^2 \theta \rangle = \langle 0, 0, 1 \rangle$.
- Dot product: $\langle 0, 0, 1 \rangle \cdot \langle \cos \theta, \sin \theta, 0 \rangle = 0$.
- The integral of zero is zero.

Answer: 0

3. Compute the flux of $\mathbf{F} = \langle x, y, 0 \rangle$ through the upper hemisphere $x^2 + y^2 + z^2 = 1, z \geq 0$, oriented upwards.

- Use the graph form $z = \sqrt{1 - x^2 - y^2}$.
- Compute partials: $z_x = -x/\sqrt{1 - x^2 - y^2}$ and $z_y = -y/\sqrt{1 - x^2 - y^2}$.
- The normal element is $\mathbf{n}dS = \langle -z_x, -z_y, 1 \rangle dx dy = \langle x/z, y/z, 1 \rangle dx dy$.
- Dot product: $\mathbf{F} \cdot \mathbf{n}dS = \langle x, y, 0 \rangle \cdot \langle x/z, y/z, 1 \rangle dx dy = (x^2/z + y^2/z) dx dy = \frac{x^2 + y^2}{z} dx dy$.
- Convert to polar: $\iint_D \frac{r^2}{\sqrt{1 - r^2}} r dr d\theta$.
- Integrate $r^3/\sqrt{1 - r^2}$ from 0 to 1 using $u = 1 - r^2$. The result is $2/3$.
- Multiply by 2π from the θ integral to get $4\pi/3$.

Answer: $4\pi/3$