

Surface integrals of scalar functions

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Practice problems

1. Compute the surface integral of $f(x, y, z) = x + y + z$ over the part of the plane $x + y + z = 1$ that lies in the first octant ($x, y, z \geq 0$).
2. Find the total mass of a thin hemispherical shell of radius $R = 2$ defined by $x^2 + y^2 + z^2 = 4, z \geq 0$, if the density is $\rho(x, y, z) = z$.
3. Calculate the average value of $f(x, y, z) = x^2 + y^2$ over the cylinder $x^2 + y^2 = 1$ for $0 \leq z \leq 2$.

Solutions

1. Compute the surface integral of $f(x, y, z) = x + y + z$ over the part of the plane $x + y + z = 1$ that lies in the first octant ($x, y, z \geq 0$).
 - Parameterise the surface as $z = 1 - x - y$ over the domain $D : 0 \leq x \leq 1, 0 \leq y \leq 1 - x$.
 - Compute the surface area element: $z_x = -1, z_y = -1$, so $dS = \sqrt{1 + (-1)^2 + (-1)^2} dx dy = \sqrt{3} dx dy$.
 - Substitute z into the function: $f = x + y + (1 - x - y) = 1$.
 - Set up the integral: $\int_0^1 \int_0^{1-x} 1 \cdot \sqrt{3} dy dx$.
 - Evaluate: $\sqrt{3} \int_0^1 (1 - x) dx = \sqrt{3} [x - x^2/2]_0^1 = \sqrt{3}/2$.

Answer: $\frac{\sqrt{3}}{2}$

2. Find the total mass of a thin hemispherical shell of radius $R = 2$ defined by $x^2 + y^2 + z^2 = 4, z \geq 0$, if the density is $\rho(x, y, z) = z$.
 - Use spherical coordinates with $\rho = 2$. The area element is $dS = R^2 \sin \phi d\theta d\phi = 4 \sin \phi d\theta d\phi$.
 - The density is $\rho = z = 2 \cos \phi$.
 - Set up the integral: $\int_0^{2\pi} \int_0^{\pi/2} (2 \cos \phi) (4 \sin \phi) d\phi d\theta$.
 - Simplify the integrand: $8 \sin \phi \cos \phi = 4 \sin(2\phi)$.
 - Evaluate: $2\pi \int_0^{\pi/2} 4 \sin(2\phi) d\phi = 2\pi [-2 \cos(2\phi)]_0^{\pi/2} = 2\pi [2 - (-2)] = 8\pi$.

Answer: 8π

3. Calculate the average value of $f(x, y, z) = x^2 + y^2$ over the cylinder $x^2 + y^2 = 1$ for $0 \leq z \leq 2$.
 - Parameterise the cylinder: $x = \cos \theta, y = \sin \theta, z = z$.
 - Compute dS : $\mathbf{r}_\theta = (-\sin \theta, \cos \theta, 0)$ and $\mathbf{r}_z = (0, 0, 1)$. The cross product is $(\cos \theta, \sin \theta, 0)$, and its magnitude is 1. So $dS = d\theta dz$.
 - Substitute into f : $f = \cos^2 \theta + \sin^2 \theta = 1$.
 - Compute the total integral: $\int_0^2 \int_0^{2\pi} 1 \cdot 1 d\theta dz = 4\pi$.
 - Compute the surface area: $\text{Area}(S) = \int_0^2 \int_0^{2\pi} 1 d\theta dz = 4\pi$.
 - Average value: $f_{\text{avg}} = 4\pi/4\pi = 1$.

Answer: 1