

Parametrised Surfaces and Surface Area

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Practice problems

1. Find the area of the part of the plane $x+y+z = 1$ that lies in the first octant ($x \geq 0, y \geq 0, z \geq 0$).
2. Calculate the area of the surface $z = xy$ over the region $0 \leq x \leq 1, 0 \leq y \leq 1$.
3. A cylinder of radius R and height H is parametrised by $\mathbf{r}(\theta, z) = \langle R \cos \theta, R \sin \theta, z \rangle$. Use the surface area formula to verify the lateral area is $2\pi RH$.

Solutions

1. Find the area of the part of the plane $x+y+z = 1$ that lies in the first octant ($x \geq 0, y \geq 0, z \geq 0$).

- Rewrite the surface as $z = 1 - x - y$.
- Compute partial derivatives: $f_x = -1$ and $f_y = -1$.
- Find the area element: $\sqrt{1 + (-1)^2 + (-1)^2} = \sqrt{3}$.
- Identify the domain D in the xy -plane: the triangle bounded by $x = 0, y = 0$, and $x + y = 1$.
- Integrate: $\int_0^1 \int_0^{1-x} \sqrt{3} dy dx = \sqrt{3} \int_0^1 (1-x) dx = \sqrt{3} [x - x^2/2]_0^1 = \sqrt{3}/2$.

Answer: $\frac{\sqrt{3}}{2}$

2. Calculate the area of the surface $z = xy$ over the region $0 \leq x \leq 1, 0 \leq y \leq 1$.

- Compute partial derivatives: $f_x = y$ and $f_y = x$.
- The area element is $\sqrt{1 + y^2 + x^2}$.
- Set up the integral: $\int_0^1 \int_0^1 \sqrt{1 + x^2 + y^2} dx dy$.

Answer: $\int_0^1 \int_0^1 \sqrt{1 + x^2 + y^2} dx dy$

3. A cylinder of radius R and height H is parametrised by $\mathbf{r}(\theta, z) = \langle R \cos \theta, R \sin \theta, z \rangle$. Use the surface area formula to verify the lateral area is $2\pi RH$.

- Compute $\mathbf{r}_\theta = \langle -R \sin \theta, R \cos \theta, 0 \rangle$.
- Compute $\mathbf{r}_z = \langle 0, 0, 1 \rangle$.
- Compute the cross product: $\mathbf{r}_\theta \times \mathbf{r}_z = \langle R \cos \theta, R \sin \theta, 0 \rangle$.
- Find the magnitude: $\sqrt{R^2 \cos^2 \theta + R^2 \sin^2 \theta + 0} = R$.
- Integrate over $0 \leq \theta \leq 2\pi$ and $0 \leq z \leq H$: $\int_0^H \int_0^{2\pi} R d\theta dz = R(2\pi)(H) = 2\pi RH$.

Answer: $2\pi RH$