

Green's theorem III — area by boundary; the planimeter

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Practice problems

1. Use the formula $\text{Area} = \frac{1}{2} \oint_C (x dy - y dx)$ to find the area of the ellipse defined by $x = 2 \cos t$ and $y = \sin t$ for $t \in [0, 2\pi]$.
2. Find the area of the region bounded by the curve $x = t^2$ and $y = t^3 - 3t$ for $t \in [-\sqrt{3}, \sqrt{3}]$.
3. A region is bounded by the line $y = x$ and the parabola $y = x^2$. Use the boundary integral $\oint_C x dy$ to find the area.

Solutions

1. Use the formula $\text{Area} = \frac{1}{2} \oint_C (x dy - y dx)$ to find the area of the ellipse defined by $x = 2 \cos t$ and $y = \sin t$ for $t \in [0, 2\pi]$.
 - Compute differentials: $dx = -2 \sin t dt$ and $dy = \cos t dt$.
 - Substitute into the integrand: $x dy - y dx = (2 \cos t)(\cos t dt) - (\sin t)(-2 \sin t dt)$.
 - Simplify the expression: $2 \cos^2 t dt + 2 \sin^2 t dt = 2(\cos^2 t + \sin^2 t) dt = 2 dt$.
 - Integrate from 0 to 2π : $\int_0^{2\pi} 2 dt = 4\pi$.
 - Multiply by the factor of $1/2$: $\text{Area} = \frac{1}{2}(4\pi) = 2\pi$.

Answer: 2π

2. Find the area of the region bounded by the curve $x = t^2$ and $y = t^3 - 3t$ for $t \in [-\sqrt{3}, \sqrt{3}]$.
 - Compute differentials: $dx = 2t dt$ and $dy = (3t^2 - 3) dt$.
 - Substitute into $x dy - y dx$: $(t^2)(3t^2 - 3) dt - (t^3 - 3t)(2t dt)$.
 - Expand and simplify: $(3t^4 - 3t^2) dt - (2t^4 - 6t^2) dt = (t^4 + 3t^2) dt$.
 - Integrate from $-\sqrt{3}$ to $\sqrt{3}$: $\int_{-\sqrt{3}}^{\sqrt{3}} (t^4 + 3t^2) dt = [\frac{1}{5}t^5 + t^3]_{-\sqrt{3}}^{\sqrt{3}}$.
 - Evaluate: $(\frac{9\sqrt{3}}{5} + 3\sqrt{3}) - (-\frac{9\sqrt{3}}{5} - 3\sqrt{3}) = \frac{18\sqrt{3}}{5} + 6\sqrt{3} = \frac{48\sqrt{3}}{5}$.
 - Multiply by $1/2$: $\text{Area} = \frac{24\sqrt{3}}{5}$.

Answer: $\frac{24\sqrt{3}}{5}$

3. A region is bounded by the line $y = x$ and the parabola $y = x^2$. Use the boundary integral $\oint_C x dy$ to find the area.
 - The boundary consists of two parts: C_1 from $(0, 0)$ to $(1, 1)$ along $y = x^2$, and C_2 from $(1, 1)$ back to $(0, 0)$ along $y = x$.
 - For C_1 : $x = t, y = t^2, dy = 2t dt$ for $t \in [0, 1]$. $\int_{C_1} x dy = \int_0^1 t(2t dt) = \int_0^1 2t^2 dt = [\frac{2}{3}t^3]_0^1 = \frac{2}{3}$.
 - For C_2 : $x = t, y = t, dy = dt$ for t from 1 to 0. $\int_{C_2} x dy = \int_1^0 t dt = [\frac{1}{2}t^2]_1^0 = 0 - \frac{1}{2} = -\frac{1}{2}$.
 - Sum the integrals: $\text{Area} = \frac{2}{3} - \frac{1}{2} = \frac{4-3}{6} = \frac{1}{6}$.

Answer: $1/6$