

Green's theorem II — proof sketch and computations

Lesson 44 · Module 4 — Green, Stokes & Divergence · Multivariable & Vector Calculus · dailymaths.uk/vectorcalc

Practice problems

1. Use Green's Theorem to evaluate $\oint_C ((y + x^2)dx + (x + y^2)dy)$ where C is the boundary of the square $[0, 1] \times [0, 1]$ oriented counter-clockwise.
2. Evaluate $\oint_C (y^2 dx + 3xy dy)$ where C is the boundary of the semi-circle $x^2 + y^2 = 1, y \geq 0$ and the segment from $(-1, 0)$ to $(1, 0)$, oriented counter-clockwise.
3. A vector field is given by $\mathbf{F} = (e^x - y, e^y + x)$. Calculate the work done by this field around the circle $x^2 + y^2 = 4$ oriented clockwise.

Solutions

1. Use Green's Theorem to evaluate $\oint_C ((y + x^2)dx + (x + y^2)dy)$ where C is the boundary of the square $[0, 1] \times [0, 1]$ oriented counter-clockwise.

- Identify $P = y + x^2$ and $Q = x + y^2$.
- Compute the 2D curl: $\frac{\partial Q}{\partial x} = 1$ and $\frac{\partial P}{\partial y} = 1$.
- The integrand is $1 - 1 = 0$.
- The double integral of 0 over the square is 0.

Answer: 0

2. Evaluate $\oint_C (y^2 dx + 3xy dy)$ where C is the boundary of the semi-circle $x^2 + y^2 = 1, y \geq 0$ and the segment from $(-1, 0)$ to $(1, 0)$, oriented counter-clockwise.

- Identify $P = y^2$ and $Q = 3xy$.
- Compute the 2D curl: $\frac{\partial Q}{\partial x} = 3y$ and $\frac{\partial P}{\partial y} = 2y$.
- The integrand is $3y - 2y = y$.
- Set up the double integral in polar coordinates: $\int_0^\pi \int_0^1 (r \sin \theta) r dr d\theta$.
- Integrate r^2 to get $1/3$ and $\sin \theta$ from 0 to π to get 2.
- The result is $2/3$.

Answer: $2/3$

3. A vector field is given by $\mathbf{F} = (e^x - y, e^y + x)$. Calculate the work done by this field around the circle $x^2 + y^2 = 4$ oriented clockwise.

- Identify $P = e^x - y$ and $Q = e^y + x$.
- Compute the 2D curl: $\frac{\partial Q}{\partial x} = 1$ and $\frac{\partial P}{\partial y} = -1$.
- The integrand is $1 - (-1) = 2$.
- The double integral over the disk of radius 2 is $2 \times \text{Area} = 2 \times (\pi \cdot 2^2) = 8\pi$.
- Since the orientation is clockwise, multiply by -1 .

Answer: -8π