Green's theorem I — statement and the boundary-interior idea made precise

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Practice problems

1. Use Green's Theorem to evaluate $\oint_C ((x+y)dx + x^2 dy)$ where C is the boundary of the square with vertices $(0,0), (1,0), (1,1), (0,1)$ oriented counter-clockwise.
2. Evaluate $\oint_C (y^2 dx + 3xy dy)$ where C is the circle $x^2 + y^2 = 4$ oriented counter-clockwise.
3. A vector field is given by $\mathbf{F} = (e^x - y)\mathbf{i} + (e^y + x)\mathbf{j}$. Find the circulation around the boundary of the region R defined by $0 \le x \le 2$ and $0 \le y \le 1$.

Solutions

1. Use Green's Theorem to evaluate $\oint_C ((x+y)dx + x^2dy)$ where C is the boundary of the square with vertices $(0,0),(1,0),(1,1),(0,1)$ oriented counter-clockwise.

   - Identify $P = x + y$ and $Q = x^2$.
   - Compute partial derivatives: $\frac{\partial Q}{\partial x} = 2x$ and $\frac{\partial P}{\partial y} = 1$.
   - The integrand is $2x - 1$.
   - Set up the double integral over the square: $\int_0^1 \int_0^1 (2x-1)dydx$.
   - Integrate with respect to y: $\int_0^1 (2x-1)dx$.
   - Integrate with respect to x: $[x^2 - x]_0^1 = (1-1) - (0-0) = 0$.

   Answer: 0

2. Evaluate $\oint_C (y^2dx + 3xydy)$ where C is the circle $x^2 + y^2 = 4$ oriented counter-clockwise.

   - Identify $P = y^2$ and $Q = 3xy$.
   - Compute partial derivatives: $\frac{\partial Q}{\partial x} = 3y$ and $\frac{\partial P}{\partial y} = 2y$.
   - The integrand is $3y - 2y = y$.
   - Set up the double integral over the disk R of radius 2: $\iint_R ydA$.
   - By symmetry, the integral of y over a disk centered at the origin is 0.

   Answer: 0

3. A vector field is given by $\mathbf{F} = (e^x - y)\mathbf{i} + (e^y + x)\mathbf{j}$. Find the circulation around the boundary of the region R defined by $0 \leq x \leq 2$ and $0 \leq y \leq 1$.

   - Identify $P = e^x - y$ and $Q = e^y + x$.
   - Compute partial derivatives: $\frac{\partial Q}{\partial x} = 1$ and $\frac{\partial P}{\partial y} = -1$.
   - The integrand is $1 - (-1) = 2$.
   - The region R is a rectangle with area $2 \times 1 = 2$.
   - The integral is $\iint_R 2dA = 2 \times \text{Area}(R) = 2 \times 2 = 4$.

   Answer: 4
