Module 3 Review + Self-Test

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Practice problems

1. Compute the work done by the vector field $\mathbf{F} = \langle 2xy, x^2 + 1 \rangle$ along the path C given by $y = x^2$ from $x = 0$ to $x = 1$.
2. Use Green's Theorem to evaluate $\oint_C (y^2 dx + 3xydy)$ where C is the boundary of the rectangle $[0, 1] \times [0, 2]$ oriented counter-clockwise.
3. Given the vector field $\mathbf{F} = \langle e^x \sin y, -e^x \cos y \rangle$, determine if the field is solenoidal, irrotational, or both.

Solutions

1. Compute the work done by the vector field $\mathbf{F} = \langle 2xy, x^2 + 1 \rangle$ along the path C given by $y = x^2$ from $x = 0$ to $x = 1$.

   - Check for conservativity: $\frac{\partial P}{\partial y} = 2x$ and $\frac{\partial Q}{\partial x} = 2x$. The field is conservative.
   - Find the potential function $f(x, y)$: $\int 2xydx = x^2y + g(y)$. Differentiating with respect to y gives $x^2 + g'(y) = x^2 + 1$, so $g'(y) = 1$ and $g(y) = y$.
   - The potential function is $f(x, y) = x^2y + y$.
   - Evaluate at endpoints: $f(1, 1) = 1^2(1) + 1 = 2$ and $f(0, 0) = 0$.
   - Work is $2 - 0 = 2$.

   Answer: 2

2. Use Green's Theorem to evaluate $\oint_C (y^2dx + 3xydy)$ where C is the boundary of the rectangle $[0, 1] \times [0, 2]$ oriented counter-clockwise.

   - Identify $P = y^2$ and $Q = 3xy$.
   - Compute the curl: $\frac{\partial Q}{\partial x} = 3y$ and $\frac{\partial P}{\partial y} = 2y$.
   - The integrand is $3y - 2y = y$.
   - Set up the double integral: $\int_0^1 \int_0^2 ydydx$.
   - Inner integral: $[\frac{1}{2}y^2]_0^2 = 2$.
   - Outer integral: $\int_0^1 2dx = 2$.

   Answer: 2

3. Given the vector field $\mathbf{F} = \langle e^x \sin y, -e^x \cos y \rangle$, determine if the field is solenoidal, irrotational, or both.

   - Compute divergence: $\frac{\partial}{\partial x}(e^x \sin y) + \frac{\partial}{\partial y}(-e^x \cos y) = e^x \sin y + e^x \sin y = 2e^x \sin y$. Not zero, so not solenoidal.
   - Compute curl: $\frac{\partial}{\partial x}(-e^x \cos y) - \frac{\partial}{\partial y}(e^x \sin y) = -e^x \cos y - e^x \cos y = -2e^x \cos y$. Not zero, so not irrotational.
   - The field is neither.

   Answer: Neither solenoidal nor irrotational
