Grad, div, curl: the identities and what they forbid

Lesson 41 · Module 3 — Vector Fields & Line Integrals · Multivariable & Vector Calculus · dailymaths.uk/vectorcalc

Practice problems

1. Determine if the vector field $\mathbf{F} = \langle 2xy + z, x^2, x \rangle$ can be written as the gradient of a scalar function f.
2. Given a vector field $\mathbf{G} = \text{curl}\,\mathbf{A}$, where $\mathbf{A} = \langle x^2, y^2, z^2 \rangle$, find the divergence of $\mathbf{G}$ without computing $\mathbf{G}$ explicitly.
3. Given a vector field $\mathbf{V} = \langle x^2 + y^2, 2xy, z^2 \rangle$, determine if it can be written as the curl of another field $\mathbf{G}$ by computing its divergence.

Solutions

1. Determine if the vector field $\mathbf{F} = \langle 2xy + z, x^2, x \rangle$ can be written as the gradient of a scalar function f.

    - Compute the curl of $\mathbf{F}$.
    - The x-component of the curl is $\frac{\partial}{\partial y}(x) - \frac{\partial}{\partial z}(x^2) = 0 - 0 = 0$.
    - The y-component of the curl is $\frac{\partial}{\partial z}(2xy + z) - \frac{\partial}{\partial x}(x) = 1 - 1 = 0$.
    - The z-component of the curl is $\frac{\partial}{\partial x}(x^2) - \frac{\partial}{\partial y}(2xy + z) = 2x - 2x = 0$.
    - Since the curl is the zero vector, the field is irrotational and can be a gradient.

    Answer: Yes, it can be a gradient because curl $\mathbf{F} = \mathbf{0}$.

2. Given a vector field $\mathbf{G} = \text{curl } \mathbf{A}$, where $\mathbf{A} = \langle x^2, y^2, z^2 \rangle$, find the divergence of $\mathbf{G}$ without computing $\mathbf{G}$ explicitly.

    - Identify that $\mathbf{G}$ is defined as the curl of another vector field $\mathbf{A}$.
    - Apply the identity $\text{div}(\text{curl } \mathbf{F}) = 0$.
    - Since $\mathbf{G} = \text{curl } \mathbf{A}$, then $\text{div } \mathbf{G} = \text{div}(\text{curl } \mathbf{A}) = 0$.

    Answer: 0

3. Given a vector field $\mathbf{V} = \langle x^2 + y^2, 2xy, z^2 \rangle$, determine if it can be written as the curl of another field $\mathbf{G}$ by computing its divergence.

    - Compute the divergence of $\mathbf{V}$.
    - $\text{div } \mathbf{V} = \frac{\partial}{\partial x}(x^2 + y^2) + \frac{\partial}{\partial y}(2xy) + \frac{\partial}{\partial z}(z^2)$.
    - $\text{div } \mathbf{V} = 2x + 2x + 2z = 4x + 2z$.
    - Since the divergence is not zero everywhere, the field is not solenoidal.
    - By the identity $\text{div}(\text{curl } \mathbf{G}) = 0$, a field must have zero divergence to be a curl.

    Answer: No, it cannot be a curl because $\text{div } \mathbf{V} \neq 0$.
