Path independence; potential functions and how to find them

Lesson 37 · Module 3 — Vector Fields & Line Integrals · Multivariable & Vector Calculus · dailymaths.uk/vectorcalc

Practice problems

1. Find the potential function $f(x, y)$ for the conservative vector field $\mathbf{F}(x, y) = (2x+y)\mathbf{i}+(x+2)\mathbf{j}$.
2. Find the potential function $f(x, y)$ for the vector field $\mathbf{F}(x, y) = (2xy + 1)\mathbf{i} + (x^2)\mathbf{j}$.
3. Find the potential function $f(x, y, z)$ for the 3D vector field $\mathbf{F} = (2xz)\mathbf{i} + (2yz)\mathbf{j} + (x^2 + y^2)\mathbf{k}$.

Solutions

1. Find the potential function $f(x, y)$ for the conservative vector field $\mathbf{F}(x, y) = (2x+y)\mathbf{i}+(x+2)\mathbf{j}$.

   - Integrate $P = 2x + y$ with respect to x: $\int(2x + y)\,dx = x^2 + xy + g(y)$.
   - Differentiate with respect to y: $\frac{\partial}{\partial y}(x^2 + xy + g(y)) = x + g'(y)$.
   - Set equal to $Q = x + 2$: $x + g'(y) = x + 2$, so $g'(y) = 2$.
   - Integrate $g'(y)$: $g(y) = 2y + C$.

   Answer: $f(x, y) = x^2 + xy + 2y + C$

2. Find the potential function $f(x, y)$ for the vector field $\mathbf{F}(x, y) = (2xy + 1)\mathbf{i} + (x^2)\mathbf{j}$.

   - Integrate $P = 2xy + 1$ with respect to x: $\int(2xy + 1)\,dx = x^2y + x + g(y)$.
   - Differentiate with respect to y: $\frac{\partial}{\partial y}(x^2y + x + g(y)) = x^2 + g'(y)$.
   - Set equal to $Q = x^2$: $x^2 + g'(y) = x^2$, so $g'(y) = 0$.
   - Integrate $g'(y)$: $g(y) = C$.

   Answer: $f(x, y) = x^2y + x + C$

3. Find the potential function $f(x, y, z)$ for the 3D vector field $\mathbf{F} = (2xz)\mathbf{i} + (2yz)\mathbf{j} + (x^2 + y^2)\mathbf{k}$.

   - Integrate $P = 2xz$ with respect to x: $f = x^2z + g(y, z)$.
   - Differentiate with respect to y: $\frac{\partial f}{\partial y} = 0 + \frac{\partial g}{\partial y}$. Set equal to $Q = 2yz$: $\frac{\partial g}{\partial y} = 2yz$.
   - Integrate $\frac{\partial g}{\partial y}$ with respect to y: $g(y, z) = y^2z + h(z)$.
   - Now $f = x^2z + y^2z + h(z)$. Differentiate with respect to z: $\frac{\partial f}{\partial z} = x^2 + y^2 + h'(z)$.
   - Set equal to $R = x^2 + y^2$: $x^2 + y^2 + h'(z) = x^2 + y^2$, so $h'(z) = 0$.

   Answer: $f(x, y, z) = (x^2 + y^2)z + C$
