The Fundamental Theorem for Line Integrals

Lesson 36 · Module 3 — Vector Fields & Line Integrals · Multivariable & Vector Calculus · dailymaths.uk/vectorcalc

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Practice problems

1. Determine if the vector field $\mathbf{F}(x, y) = (2xy + 1)\mathbf{i} + (x^2 + 3y^2)\mathbf{j}$ is conservative. If so, find the work done moving from $(0, 0)$ to $(1, 2)$.
2. A particle moves in a conservative field with potential $f(x, y, z) = e^{x+y} + z^2$. Calculate the work done moving from $(0, 0, 0)$ to $(1, 0, 2)$.
3. Show that the work done by the field $\mathbf{F}(x, y) = (y^2)\mathbf{i} + (2xy)\mathbf{j}$ around the unit circle $x^2 + y^2 = 1$ is zero.

Solutions

1. Determine if the vector field $\mathbf{F}(x, y) = (2xy + 1)\mathbf{i} + (x^2 + 3y^2)\mathbf{j}$ is conservative. If so, find the work done moving from $(0, 0)$ to $(1, 2)$.

   - Check conservativeness: $P_y = \frac{\partial}{\partial y}(2xy + 1) = 2x$ and $Q_x = \frac{\partial}{\partial x}(x^2 + 3y^2) = 2x$. Since $P_y = Q_x$, it is conservative.
   - Find potential f: $\int(2xy + 1)dx = x^2y + x + g(y)$.
   - Differentiate with respect to y: $x^2 + g'(y) = x^2 + 3y^2$, so $g'(y) = 3y^2$, which means $g(y) = y^3$.
   - Potential function is $f(x, y) = x^2y + x + y^3$.
   - Compute $f(1, 2) - f(0, 0) = (1^2(2) + 1 + 2^3) - 0 = 2 + 1 + 8 = 11$.

   Answer: 11

2. A particle moves in a conservative field with potential $f(x, y, z) = e^{x+y} + z^2$. Calculate the work done moving from $(0, 0, 0)$ to $(1, 0, 2)$.

   - Since the potential f is given, we use the Fundamental Theorem directly.
   - Evaluate f at the end point $(1, 0, 2)$: $f(1, 0, 2) = e^{1+0} + 2^2 = e + 4$.
   - Evaluate f at the start point $(0, 0, 0)$: $f(0, 0, 0) = e^{0+0} + 0^2 = 1$.
   - Work $W = f(1, 0, 2) - f(0, 0, 0) = e + 4 - 1 = e + 3$.

   Answer: e + 3

3. Show that the work done by the field $\mathbf{F}(x, y) = (y^2)\mathbf{i} + (2xy)\mathbf{j}$ around the unit circle $x^2 + y^2 = 1$ is zero.

   - Check if $\mathbf{F}$ is conservative: $P = y^2$, so $P_y = 2y$. $Q = 2xy$, so $Q_x = 2y$.
   - Since $P_y = Q_x$, the field is conservative.
   - The unit circle is a closed loop, meaning the start and end points are identical.
   - By the Fundamental Theorem for Line Integrals, the integral of a conservative field over any closed loop is zero.

   Answer: 0
