Line integrals of vector fields — work along a path

Lesson 35 · Module 3 — Vector Fields & Line Integrals · Multivariable & Vector Calculus · dailymaths.uk/vectorcalc

Practice problems

1. Compute the work done by the vector field $\mathbf{F}(x, y) = 2x\mathbf{i} + 3y\mathbf{j}$ along the straight line segment from $(0, 0)$ to $(2, 1)$.
2. Find the work done by the field $\mathbf{F}(x, y) = -y\mathbf{i} + x\mathbf{j}$ along the upper semi-circle of radius 2 from $(2, 0)$ to $(-2, 0)$.
3. A particle moves along the path $\mathbf{r}(t) = t^2\mathbf{i} + t^3\mathbf{j}$ from $t = 0$ to $t = 1$. The force field is $\mathbf{F}(x, y) = y\mathbf{i} - x\mathbf{j}$. Calculate the total work done.

Solutions

1. Compute the work done by the vector field $\mathbf{F}(x, y) = 2x\mathbf{i} + 3y\mathbf{j}$ along the straight line segment from $(0, 0)$ to $(2, 1)$.

    - Parameterize the path: $\mathbf{r}(t) = 2t\mathbf{i} + t\mathbf{j}$ for $t \in [0, 1]$.
    - Find the velocity vector: $\mathbf{r}'(t) = 2\mathbf{i} + 1\mathbf{j}$.
    - Substitute the path into the field: $\mathbf{F}(\mathbf{r}(t)) = 2(2t)\mathbf{i} + 3(t)\mathbf{j} = 4t\mathbf{i} + 3t\mathbf{j}$.
    - Compute the dot product: $\mathbf{F} \cdot \mathbf{r}' = (4t)(2) + (3t)(1) = 8t + 3t = 11t$.
    - Integrate from 0 to 1: $\int_0^1 11t\,dt = [5.5t^2]_0^1 = 5.5$.

    Answer: 5.5

2. Find the work done by the field $\mathbf{F}(x, y) = -y\mathbf{i} + x\mathbf{j}$ along the upper semi-circle of radius 2 from $(2, 0)$ to $(-2, 0)$.

    - Parameterize the path: $\mathbf{r}(t) = 2\cos(t)\mathbf{i} + 2\sin(t)\mathbf{j}$ for $t \in [0, \pi]$.
    - Find the velocity vector: $\mathbf{r}'(t) = -2\sin(t)\mathbf{i} + 2\cos(t)\mathbf{j}$.
    - Substitute into the field: $\mathbf{F}(\mathbf{r}(t)) = -2\sin(t)\mathbf{i} + 2\cos(t)\mathbf{j}$.
    - Compute the dot product: $\mathbf{F} \cdot \mathbf{r}' = (-2\sin t)(-2\sin t) + (2\cos t)(2\cos t) = 4\sin^2 t + 4\cos^2 t = 4$.
    - Integrate from 0 to π: $\int_0^\pi 4\,dt = [4t]_0^\pi = 4\pi$.

    Answer: 4\pi

3. A particle moves along the path $\mathbf{r}(t) = t^2\mathbf{i} + t^3\mathbf{j}$ from $t = 0$ to $t = 1$. The force field is $\mathbf{F}(x, y) = y\mathbf{i} - x\mathbf{j}$. Calculate the total work done.

    - Find the velocity vector: $\mathbf{r}'(t) = 2t\mathbf{i} + 3t^2\mathbf{j}$.
    - Substitute the path into the field: $\mathbf{F}(\mathbf{r}(t)) = t^3\mathbf{i} - t^2\mathbf{j}$.
    - Compute the dot product: $\mathbf{F} \cdot \mathbf{r}' = (t^3)(2t) + (-t^2)(3t^2) = 2t^4 - 3t^4 = -t^4$.
    - Integrate from 0 to 1: $\int_0^1 -t^4\,dt = [-\frac{1}{5}t^5]_0^1 = -0.2$.

    Answer: -0.2
