Line integrals of scalar functions — mass of a wire

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Practice problems

1. Find the mass of a wire along the line segment from $(0, 0)$ to $(2, 3)$ if the density is given by $f(x, y) = x^2 + y$.
2. Compute the line integral $\int_C x\,ds$ where C is the quarter circle $x^2 + y^2 = 4$ in the first quadrant.
3. A wire follows the path $\mathbf{r}(t) = \langle t, t^2, \frac{2}{3}t^3 \rangle$ for $0 \leq t \leq 1$. If the density is $f(x, y, z) = 1$, find the total mass (which is the arc length).

Solutions

1. Find the mass of a wire along the line segment from $(0, 0)$ to $(2, 3)$ if the density is given by $f(x, y) = x^2 + y$.

   - Parameterize the line: $\mathbf{r}(t) = \langle 2t, 3t\rangle$ for $0 \leq t \leq 1$.
   - Compute the derivative: $\mathbf{r}'(t) = \langle 2, 3\rangle$.
   - Find the speed: $|\mathbf{r}'(t)| = \sqrt{2^2 + 3^2} = \sqrt{13}$.
   - Substitute into the density function: $f(2t, 3t) = (2t)^2 + 3t = 4t^2 + 3t$.
   - Set up the integral: $\int_0^1 (4t^2 + 3t)\sqrt{13}\, dt$.
   - Integrate: $\sqrt{13}[\frac{4}{3}t^3 + \frac{3}{2}t^2]_0^1 = \sqrt{13}(\frac{4}{3} + \frac{3}{2}) = \sqrt{13}(\frac{17}{6})$.

   Answer: \frac17\sqrt136

2. Compute the line integral $\int_C x\, ds$ where C is the quarter circle $x^2 + y^2 = 4$ in the first quadrant.

   - Parameterize the curve: $\mathbf{r}(t) = \langle 2\cos t, 2\sin t\rangle$ for $0 \leq t \leq \pi/2$.
   - Compute the derivative: $\mathbf{r}'(t) = \langle -2\sin t, 2\cos t\rangle$.
   - Find the speed: $|\mathbf{r}'(t)| = \sqrt{4\sin^2 t + 4\cos^2 t} = 2$.
   - Substitute into the integrand: $x = 2\cos t$.
   - Set up the integral: $\int_0^{\pi/2} (2\cos t)(2)\, dt = 4\int_0^{\pi/2} \cos t\, dt$.
   - Evaluate: $4[\sin t]_0^{\pi/2} = 4(1 - 0) = 4$.

   Answer: 4

3. A wire follows the path $\mathbf{r}(t) = \langle t, t^2, \frac{2}{3}t^3\rangle$ for $0 \leq t \leq 1$. If the density is $f(x, y, z) = 1$, find the total mass (which is the arc length).

   - Compute the derivative: $\mathbf{r}'(t) = \langle 1, 2t, 2t^2\rangle$.
   - Find the speed: $|\mathbf{r}'(t)| = \sqrt{1^2 + (2t)^2 + (2t^2)^2} = \sqrt{1 + 4t^2 + 4t^4}$.
   - Recognize the perfect square: $\sqrt{(1 + 2t^2)^2} = 1 + 2t^2$.
   - Set up the integral: $\int_0^1 (1 + 2t^2)\, dt$.
   - Integrate: $[t + \frac{2}{3}t^3]_0^1 = 1 + \frac{2}{3} = \frac{5}{3}$.

   Answer: \frac53
