Module 2 review + self-test

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Practice problems

1. Evaluate the integral $\int_0^1 \int_y^1 \sin(x^2)\, dx\, dy$ by swapping the order of integration.
2. Find the volume of the solid bounded by the cylinder $x^2 + y^2 = 4$ and the planes $z = 0$ and $z = x + 3$.
3. Use a change of variables to evaluate $\iint_R (x+y)^2 e^{x-y}\, dA$ where R is the square with vertices $(1,0), (2,1), (1,2), (0,1)$.

Solutions

1. Evaluate the integral $\int_0^1 \int_y^1 \sin(x^2)\, dx\, dy$ by swapping the order of integration.

   - Sketch the region: y goes from 0 to 1, and x goes from y to 1. This is a triangle with vertices $(0,0),(1,0),(1,1)$.
   - Swap the order: x now goes from 0 to 1, and for a fixed x, y goes from 0 to x.
   - Set up the new integral: $\int_0^1 \int_0^x \sin(x^2)\, dy\, dx$.
   - Integrate with respect to y: $\int_0^1 [y\sin(x^2)]_0^x\, dx = \int_0^1 x\sin(x^2)\, dx$.
   - Use u-substitution $u = x^2$, $du = 2xdx$: $[-\frac{1}{2}\cos(x^2)]_0^1 = \frac{1}{2}(1-\cos(1))$.

   Answer: $\frac{1}{2}(1-\cos(1))$

2. Find the volume of the solid bounded by the cylinder $x^2 + y^2 = 4$ and the planes $z = 0$ and $z = x + 3$.

   - Use cylindrical coordinates: $x = r\cos\theta, y = r\sin\theta, z = z$.
   - The region is $0 \le r \le 2, 0 \le \theta \le 2\pi, 0 \le z \le r\cos\theta + 3$.
   - Set up the integral: $\int_0^{2\pi} \int_0^2 \int_0^{r\cos\theta+3} r\, dz\, dr\, d\theta$.
   - Integrate z: $\int_0^{2\pi} \int_0^2 r(r\cos\theta + 3)\, dr\, d\theta = \int_0^{2\pi} \int_0^2 (r^2\cos\theta + 3r)\, dr\, d\theta$.
   - Integrate r: $\int_0^{2\pi} [\frac{1}{3}r^3\cos\theta + \frac{3}{2}r^2]_0^2\, d\theta = \int_0^{2\pi} (\frac{8}{3}\cos\theta + 6)\, d\theta$.
   - Integrate θ: $[\frac{8}{3}\sin\theta + 6\theta]_0^{2\pi} = 12\pi$.

   Answer: 12π

3. Use a change of variables to evaluate $\iint_R (x+y)^2 e^{x-y}\, dA$ where R is the square with vertices $(1,0),(2,1),(1,2),(0,1)$.

   - Define $u = x + y$ and $v = x - y$.
   - The vertices in (u,v) are: $(1,0) \to (1,1), (2,1) \to (3,1), (1,2) \to (3,-1), (0,1) \to (1,-1)$.
   - The region S is $1 \le u \le 3$ and $-1 \le v \le 1$.
   - Compute Jacobian: $x = (u+v)/2, y = (u-v)/2$. $\det J = 1/2$.
   - Set up integral: $\int_{-1}^1 \int_1^3 u^2 e^v \cdot \frac{1}{2}\, du\, dv$.
   - Integrate u: $\frac{1}{2}\int_{-1}^1 e^v [\frac{1}{3}u^3]_1^3\, dv = \frac{1}{2}\int_{-1}^1 e^v (9 - 1/3)\, dv = \frac{1}{2} \cdot \frac{26}{3} \int_{-1}^1 e^v\, dv$.
   - Integrate v: $\frac{13}{3}[e^v]_{-1}^1 = \frac{13}{3}(e - e^{-1})$.

   Answer: $\frac{13}{3}(e - e^{-1})$