Integration gauntlet — choosing coordinates wisely

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Practice problems

1. Evaluate the integral of $x^2 + y^2$ over the region R in the first quadrant bounded by $x^2 + y^2 = 1$, $x = 0$, and $y = 0$.
2. Find the volume of the region bounded by the cone $z = \sqrt{x^2 + y^2}$ and the sphere $x^2 + y^2 + z^2 = 1$ in the first octant.
3. Evaluate $\iint_R (x + y)^2 e^{x-y} \, dA$ where R is the square with vertices $(0, 0), (1, 1), (2, 0), (1, -1)$.

Solutions

1. Evaluate the integral of $x^2 + y^2$ over the region R in the first quadrant bounded by $x^2 + y^2 = 1$, $x = 0$, and $y = 0$.

   - Recognize the region is a quarter disk of radius 1, suggesting polar coordinates.
   - Substitute $x^2 + y^2 = r^2$ and $dA = r\,dr\,d\theta$.
   - Set bounds: r from 0 to 1, θ from 0 to $\pi/2$.
   - Compute $\int_0^{\pi/2} \int_0^1 r^2 \cdot r\,dr\,d\theta = \int_0^{\pi/2} [r^4/4]_0^1\,d\theta = \int_0^{\pi/2} 1/4\,d\theta = \pi/8$.

   Answer: \pi/8

2. Find the volume of the region bounded by the cone $z = \sqrt{x^2 + y^2}$ and the sphere $x^2+y^2+z^2 = 1$ in the first octant.

   - The region is a spherical sector, suggesting spherical coordinates.
   - The cone $z = \sqrt{x^2 + y^2}$ corresponds to $\phi = \pi/4$.
   - The sphere $x^2 + y^2 + z^2 = 1$ corresponds to $\rho = 1$.
   - First octant means θ goes from 0 to $\pi/2$.
   - Compute $\int_0^{\pi/2} \int_0^{\pi/4} \int_0^1 \rho^2 \sin\phi\,d\rho\,d\phi\,d\theta$.
   - $\int_0^1 \rho^2\,d\rho = 1/3$. $\int_0^{\pi/4} \sin\phi\,d\phi = [-\cos\phi]_0^{\pi/4} = 1 - \sqrt{2}/2$.
   - Total volume is $(\pi/2) \cdot (1/3) \cdot (1 - \sqrt{2}/2) = \frac{\pi(2-\sqrt{2})}{12}$.

   Answer: \frac\pi(2-\sqrt2)12

3. Evaluate $\iint_R (x + y)^2 e^{x-y}\,dA$ where R is the square with vertices $(0, 0), (1, 1), (2, 0), (1, -1)$.

   - The region is a rotated square. Define $u = x + y$ and $v = x - y$.
   - The vertices in uv coordinates are $(0, 0), (2, 0), (2, 2), (0, 2)$. This is a square $0 \le u \le 2, 0 \le v \le 2$.
   - Compute the Jacobian: $x = (u+v)/2, y = (u-v)/2$. $\det J = |(1/2)(1/2) - (1/2)(-1/2)| = 1/2$.
   - The integral becomes $\int_0^2 \int_0^2 u^2 e^v (1/2)\,du\,dv$.
   - $\int_0^2 u^2\,du = 8/3$. $\int_0^2 e^v\,dv = e^2 - 1$.
   - Result is $(1/2) \cdot (8/3) \cdot (e^2 - 1) = \frac{4}{3}(e^2 - 1)$.

   Answer: \frac43(e^2 - 1)
