

Change of variables clinic

Lesson 30 · Module 2 — Multiple Integrals · Multivariable & Vector Calculus · dailymaths.uk/vectorcalc

Practice problems

1. Find the area of the region bounded by the ellipse $\frac{x^2}{16} + \frac{y^2}{1} = 1$.
2. Evaluate $\iint_R (x+y) \, dA$ where R is the region bounded by $x+y=0$, $x+y=2$, $x-y=0$, $x-y=2$.
3. Compute the integral $\iint_{\mathbb{R}^2} e^{-2(x^2+y^2)} \, dA$.

Solutions

1. Find the area of the region bounded by the ellipse $\frac{x^2}{16} + \frac{y^2}{1} = 1$.

- Identify semi axes $a = 4$ and $b = 1$.
- Use transformation $x = 4r \cos \theta, y = r \sin \theta$.
- Compute Jacobian $\det J = (4)(1)r = 4r$.
- Set up integral $\int_0^{2\pi} \int_0^1 4r \, dr \, d\theta$.
- Integrate $4r$ to get $2r^2$, evaluated from 0 to 1 is 2.
- Integrate 2 from 0 to 2π to get 4π .

Answer: 4π

2. Evaluate $\iint_R (x+y) \, dA$ where R is the region bounded by $x+y = 0, x+y = 2, x-y = 0, x-y = 2$.

- Let $u = x + y$ and $v = x - y$.
- The bounds are $u \in [0, 2]$ and $v \in [0, 2]$.
- The Jacobian absolute value is $|\det J| = 1/2$.
- The integrand $x + y$ becomes u .
- Set up integral $\int_0^2 \int_0^2 u \cdot (1/2) \, du \, dv$.
- Inner integral $\int_0^2 (u/2) \, du = [u^2/4]_0^2 = 1$.
- Outer integral $\int_0^2 1 \, dv = 2$.

Answer: 2

3. Compute the integral $\iint_{\mathbb{R}^2} e^{-2(x^2+y^2)} \, dA$.

- Switch to polar coordinates: $x^2 + y^2 = r^2$ and $dA = r \, dr \, d\theta$.
- Set up integral $\int_0^{2\pi} \int_0^\infty e^{-2r^2} r \, dr \, d\theta$.
- Use substitution $w = 2r^2, dw = 4r \, dr$, so $r \, dr = dw/4$.
- Inner integral $\int_0^\infty e^{-w} (1/4) \, dw = [-1/4 e^{-w}]_0^\infty = 1/4$.
- Outer integral $\int_0^{2\pi} (1/4) \, d\theta = 2\pi/4 = \pi/2$.

Answer: $\pi/2$