The Jacobian and change of variables — the grid-warp returns

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Practice problems

1. Evaluate $\iint_R (x^2 + y^2)\, dA$ where R is the region bounded by $x + y = 1$, $x + y = 3$, $x - y = 0$, and $x - y = 2$.
2. Find the Jacobian determinant for the transformation $x = u^2 - v^2$ and $y = 2uv$.
3. Use a change of variables to evaluate $\iint_R e^{x-y}\, dA$ where R is the region bounded by $x - y = 0$, $x - y = 1$, $x + y = 0$, and $x + y = 2$.

Solutions

1. Evaluate $\iint_R (x^2 + y^2)\, dA$ where R is the region bounded by $x + y = 1$, $x + y = 3$, $x - y = 0$, and $x - y = 2$.

   - Let $u = x + y$ and $v = x - y$. The bounds are $1 \leq u \leq 3$ and $0 \leq v \leq 2$.
   - Solve for x and y: $x = (u + v)/2$ and $y = (u - v)/2$.
   - Compute the Jacobian: $\det J = \det \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{pmatrix} = -1/2$. The absolute value is 1/2.
   - Substitute into the integrand: $x^2 + y^2 = ((u + v)/2)^2 + ((u - v)/2)^2 = (u^2 + 2uv + v^2 + u^2 - 2uv + v^2)/4 = (2u^2 + 2v^2)/4 = (u^2 + v^2)/2$.
   - Set up the integral: $\int_0^2 \int_1^3 \frac{u^2+v^2}{2} \cdot \frac{1}{2}\, du\, dv = \frac{1}{4} \int_0^2 [\frac{u^3}{3} + v^2 u]_1^3\, dv$.
   - Evaluate: $\frac{1}{4} \int_0^2 (\frac{26}{3} + 2v^2)\, dv = \frac{1}{4}[\frac{26}{3}v + \frac{2v^3}{3}]_0^2 = \frac{1}{4}(\frac{52}{3} + \frac{16}{3}) = \frac{1}{4} \cdot \frac{68}{3} = \frac{17}{3}$.

   Answer: 17/3

2. Find the Jacobian determinant for the transformation $x = u^2 - v^2$ and $y = 2uv$.

   - Compute partials of x: $\frac{\partial x}{\partial u} = 2u$, $\frac{\partial x}{\partial v} = -2v$.
   - Compute partials of y: $\frac{\partial y}{\partial u} = 2v$, $\frac{\partial y}{\partial v} = 2u$.
   - Set up the determinant: $\det J = (2u)(2u) - (-2v)(2v)$.
   - Simplify: $4u^2 + 4v^2 = 4(u^2 + v^2)$.

   Answer: 4(u^2 + v^2)

3. Use a change of variables to evaluate $\iint_R e^{x-y}\, dA$ where R is the region bounded by $x - y = 0$, $x - y = 1$, $x + y = 0$, and $x + y = 2$.

   - Let $u = x - y$ and $v = x + y$. The bounds are $0 \leq u \leq 1$ and $0 \leq v \leq 2$.
   - The Jacobian for this transformation is 1/2 (as calculated in the lesson example).
   - The integral becomes $\int_0^2 \int_0^1 e^u \cdot \frac{1}{2}\, du\, dv$.
   - Integrate e^u: $\frac{1}{2} \int_0^2 [e^u]_0^1\, dv = \frac{1}{2} \int_0^2 (e - 1)\, dv$.
   - Integrate with respect to v: $\frac{1}{2}(e - 1)[v]_0^2 = \frac{1}{2}(e - 1) \cdot 2 = e - 1$.

   Answer: e - 1
