Spherical coordinates in action

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Practice problems

1. Find the mass of a hemisphere of radius R with constant density ρ_0.
2. Calculate the volume of the region inside the sphere $\rho = 4$ and the cone $\phi = \pi/6$.
3. Evaluate the integral of $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$ over the full sphere of radius 1.

Solutions

1. Find the mass of a hemisphere of radius R with constant density ρ_0.
   - Set up the integral in spherical coordinates: $M = \int_0^{2\pi} \int_0^{\pi/2} \int_0^R \rho_0 \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$.
   - Integrate ρ^2 to get $R^3/3$.
   - Integrate $\sin\phi$ from 0 to $\pi/2$ to get 1.
   - Integrate $d\theta$ from 0 to 2π to get 2π.
   - Multiply results: $\rho_0 \cdot (R^3/3) \cdot 1 \cdot 2\pi$.

   Answer: M = \frac23\pi \rho_0 R^3

2. Calculate the volume of the region inside the sphere $\rho = 4$ and the cone $\phi = \pi/6$.
   - Set up the integral: $V = \int_0^{2\pi} \int_0^{\pi/6} \int_0^4 \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$.
   - The ρ integral is $4^3/3 = 64/3$.
   - The ϕ integral is $[-\cos\phi]_0^{\pi/6} = 1 - \sqrt{3}/2$.
   - The θ integral is 2π.
   - Combine: $2\pi \cdot (1 - \sqrt{3}/2) \cdot 64/3$.

   Answer: V = \frac64\pi3(2 - \sqrt3)

3. Evaluate the integral of $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$ over the full sphere of radius 1.
   - Convert the function to spherical: $f = \rho$.
   - Include the Jacobian: $\rho \cdot \rho^2 \sin\phi = \rho^3 \sin\phi$.
   - Set bounds: $\rho \in [0, 1], \phi \in [0, \pi], \theta \in [0, 2\pi]$.
   - Integrate ρ^3 to get $1/4$.
   - Integrate $\sin\phi$ to get 2.
   - Integrate $d\theta$ to get 2π.
   - Multiply: $(1/4) \cdot 2 \cdot 2\pi$.

   Answer: \pi
